The Coordination of Intermediation*

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Abstract

We study endogenous trading among financial intermediaries, the implied intermediation chains, and market fragility in a dynamic model of intermediated asset market. We show that the endogenous emergence of an inter-dealer market affects the coordination motives in dealers' liquidity provision decisions. In an equilibrium where the inter-dealer market is active (inactive), the intermediation chain is longer (shorter), dealers hold a higher (lower) inventory on average, and they provide more (less) liquidity. Importantly, multiple equilibria may arise, suggesting market fragility in the sense of a shutdown of trading among intermediaries and liquidity drop even without a fundamental shock.

KEYWORDS: intermediation, inter-dealer trading, liquidity, multiple equilibria, fragility.

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1 Introduction

Many over-the-counter (OTC) asset markets are intermediated in the sense that buyers and sellers cannot transact with each other directly but only through an intermediary, or equivalently, a dealer. Most intermediaries do not only stand as middlemen but also transacts with each other bilaterally in an OTC inter-dealer market, leading to intermediation chains between between buyers and sellers.[1]

Yet, there is a long-standing debate regarding the bright and dark sides of trading among intermediaries and the implied intermediation chains. It is well known that trading among intermediaries allows them to manage their inventory more efficiently (Ho and Stoll, 1983, Viswanathan and Wang, 2004). But a longer intermediation chain implied by inter-dealer trading also leads to higher intermediation costs (Philippon, 2015) and more fragility due to the spread of shocks, that is, contagion (Allen and Gale, 2000). These issues have triggered critical debates regarding the role of trading among intermediaries as well as scrutiny by key regulators.

In this paper, we offer a new strategic perspective to look into trading among intermediaries and its implications on market fragility. We specifically define market fragility as the possibility of a liquidity drop in an asset market even without a fundamental shock. Somewhat in contrast to the conventional wisdom, we show that market fragility decreases rather than increases when it is easier for intermediaries to trade with each other and to form longer intermediation chains.

In doing so, we formulate the endogenous emergence of the inter-dealer market and its potential fragility in a dynamic, search-based intermediated asset market à la Rubinstein and Wolinsky (1987) and Duffie, Garleanu and Pedersen (2005). In our model, buyers and sellers can only transact through dealers, and dealers' participation and inventory holding decisions are both endogenous. At any given point of time, a fixed mass of buyers and sellers, that is, customers, arrive in the market looking for buying and selling an unit of identical asset. If successfully meets a dealer, a customer negotiates a price with the dealer through bargaining, completes the transaction, and leaves the market. But some customers may also be forced to leave the market before meeting a dealer, capturing a potential default risk. This implies that equilibrium asset illiquidity in this market is naturally measured by the mass of actively searching buyers and sellers.

[1] Many major OTC asset markets feature an OTC inter-dealer market and intermediation chains, for example, corporate bonds (Di Maggio, Kermani and Song, 2017), municipal bonds (Green, Hollifield and Schurhoff, 2006), and asset-backed securities (Hollifield, Neklyudov and Spatt, 2016). In addition, inter-dealer markets have long been recognized as important in equity markets (Hansch, Naik and Viswanathan, 1998, Reiss and Werner, 1998).

In this model, an inter-dealer market may endogenously emerge (or not) in equilibrium. When dealers' inventory cost is sufficiently low (high) relative to the potential gain from trade, dealers are more (less) willing to intermediate order flows between buyers and sellers, implying an higher (lower) inventory holding on average and in particular a larger (smaller) dispersion of inventory distribution among dealers. Since the gains from a potential inter-dealer trade are positive (negative) when the dispersion of inventory distribution is sufficiently large (small), the inter-dealer market becomes active (inactive), and dealers ultimately provide more (less) liquidity in equilibrium.

Our paper goes deeper by examining the more interesting case when dealers' inventory cost (relative to the potential gain from trade) is within an intermediate range, which is more relevant empirically. In this case, we show that the potential for forming the inter-dealer market significantly affects strategic interactions in dealers' inventory holding and liquidity provision decisions, which may lead to multiple equilibria, and, eventually, market fragility.

To understand how multiple equilibria and market fragility happen, we explicitly illustrate the strategic complementarity in our economy, that is, the coordination motives, among dealers' inventory holding and liquidity provision decisions.

First, very intuitively, inventory cost limits a dealer' ability to provide liquidity, that is, take orders from the buyers and sellers. In particular, without an active inter-dealer market, a dealer can only transact with buyers and sellers but not with other dealers. Thus, the only way for an inventory-holding dealer to offload its inventory is to meet and transact with an actively searching buyer. This difficulty in offloading inventory holdings limits a dealer's inventory management capacity, discouraging the dealer's willingness to take an inflow order from an actively searching seller to its balance sheet in the first place.

However, importantly in our framework, the competition among dealers implies that, a dealer will be more willing to hold a higher level of inventory (despite the difficulty in terms of inventory management) if other dealers hold a higher level of inventory and provide more liquidity: a strategic complementarity. To see this, notice that more customer order flows will be taken by the intermediation sector when other dealers are all willing to hold a higher level of inventory and provide more liquidity. This immediately implies that, the mass of customers who are actively searching, that is, waiting to be served, declines. Because dealers are identical and they compete with each other for order flows, it becomes harder for any given dealer to meet a customer in the

first place. Thus, this given dealer is more willing to hold a higher level of inventory itself and provide more liquidity in order to generate a higher profit per a chance of meeting a customer. An alternative way to see this perhaps surprising result is the following. In an intermediated asset market, a dealer's profit essentially comes from the asset being hard to trade, in our model reflected by customers actively searching and thus waiting in equilibrium. When fewer customers are waiting, the room to make a profit for a given dealer naturally shrinks, which forces the dealer to take an otherwise unwanted sell order into its inventory. This strategic complementarity suggests potential coordination failures in dealers' liquidity provision decisions.

The key message of our paper is that, an active inter-dealer market and the resulting intermediation chains, if endogenously emerged, can mitigate or even eliminate the concern of coordination failures. Specifically, an active inter-dealer market allows an high-inventory-holding dealer to transact with a low-inventory-holding dealer as they meet, giving high-inventory-holding dealers an alternative way to offload its costly inventory. Hence, this encourages a dealer to take a sell order to its balance sheet in the first place. Importantly, we show that, if the friction of inter-dealer trading is sufficiently small in the sense that it is sufficiently easy for dealers to meet and transact with each other, an endogenously emerged inter-dealer market can completely eliminate the strategic complementarity among dealers' inventory holding, thereby eliminate any possible coordination failures which would have happened without the inter-dealer market. On the flip side, as the inter-dealer market friction becomes higher, it is more likely for the strategic complementarity to emerge even if the inter-dealer market endogenously emerges, implying a higher possibility of coordination failures among dealers.

Regarding asset pricing implications, we show that the equilibrium with (without) an active inter-dealer market, equivalently, with a longer (shorter) intermediation chain, always features higher (lower) liquidity for a fixed economic environment. When equilibrium multiplicity happens, a switch between the two equilibria may result from non-fundamental reasons, Thus, our model points to the possibility of a relatively abrupt shutdown of the inter-dealer market and consequently a liquidity drop, even if dealer inventory cost does not change drastically. This corresponds to our earlier definition of market fragility. Equilibrium multiplicity result suggests potentially large adverse effects on asset market liquidity of some certain post-crisis regulatory policies aiming for

reducing financial stability risks but effectively increased intermediaries' inventory costs.[2] This is also consistent with various recent studies that document the retreat of inter-dealer trading in corporate bond markets (Bessembinder, Maxwell, Jacobsen and Venkataraman, 2017, Choi and Huh, 2017, Schultz, 2017). Since the financial crisis, many OTC markets have also witnessed retreats of trading activities among intermediaries and flash-crash-type instability in the absence of a major fundamental shock. For example, inter-dealer positions in the global credit default swap (CDS) markets has declined from $18 trillion in mid-2011 to $2 trillion in 2017, accompanied by a significant liquidity drop in the single-name CDS markets.[3] Overall, our results suggest that inter-dealer markets and the implied intermediation chains are beneficial to asset market liquidity and stability, and the effects work through mitigating dealers' coordination failures in inventory management. Thus, our paper provides a new perspective to reconcile the debates concerns the costs and benefits of trading among intermediaries.

Our model generates empirically plausible implications along many other dimensions. For example, as inventory cost is always considered relative to the potential gain from trade, that is, customers' trading needs, our model also predicts that the inter-dealer market is more (less) likely to emerge when customers' trading needs are higher (lower). When customers' trading needs are intermediate, multiplicity may arise. This is consistent with recent empirical patterns that the inter-dealer markets tend to provide relatively more liquidity in crisis times but they may also be fragile in these times (Di Maggio, Kermani and Song, 2017, Hollifield, Neklyudov and Spatt, 2016).

Related literature. The main contribution of our paper is to introduce and formulate the notion of coordination and market fragility into a full-fledged search-based dynamic asset pricing model. By doing so, we offer a new economic rationale to the emergence and potential fragility of trading between intermediaries. Methodologically, our paper is thus related to a macroeconomic literature focusing on coordination failures in firms' dynamic entry and exit decisions (see Cooper and John, 1988). In a similar spirit but with a different focus, Di Maggio (2016) considers the disruptive coordination of speculators in financial markets with a focus on the amplification of uncertainty risks. Also related is He and Milbradt (2014) who consider the interaction between endogenous bond default and secondary market liquidity in an OTC search framework, but without

[2] For example, the Volcker Rule and the Supplementary Leverage Ratio. See Duffie (2016) and Greenwood, Hanson, Stein, and Sunderam (2017) for more detailed descriptions and assessments of these policies.

[3] See "International Banking and Financial Market Developments," *Bank for International Settlements Quarterly Review*, June 2018.

focusing on multiple equilibria or the notion of fragility.

Our paper is closely related to a few recent papers that highlight the bright side of intermediation chains. Glode and Opp (2016) argue that longer intermediation chains between a buyer and a seller with market power can help mitigate adverse selection and lead to more efficient trading outcomes. Babus and Hu (2016) argue that intermediation chains and the resulting trading networks can help incentivize better monitoring. Ours differs by focusing on the effects on market fragility and suggests that more efficient inter-dealer trading and the resulting longer intermediation chains can sustain more stable liquidity provision.

More broadly, our paper also contributes to a burgeoning literature exploring the endogenous emergence of market structures in OTC markets. Several papers focus on the emergence of the core-periphery trading networks observed in various OTC markets, asking why some traders become customers while others become dealers. These models start either from ex-ante identical traders (Wang, 2017) or heterogenous traders along the dimension of initial asset positions (Afonso and Lagos, 2015), trading and liquidity needs (Atkeson, Eisfeldt and Weill, 2015, Neklyudov and Sambalaibat, 2015, Chang and Zhang, 2016, Hugonnier, Lester and Weill, 2016), trading technologies (Neklyudov, 2014, Farboodi, Jarosch and Shimer, 2017), or both trading needs and technologies (Uslu, 2017). Our work complement them in the sense that we have a particular focus on the emergence and fragility of the inter-dealer market and the implied intermediation chains, highlighting the underlying coordination motives of intermediation and the potential for multiple equilibria. In doing so, we fix the roles of dealers and customers but otherwise use a rich setting where the dealers are ex-ante identical and the pricing and inventory holding decisions of market participants are all endogenous. We also relax the commonly used 0-1 asset holding restrictions in many papers of this literature (with Afonso and Lagos 2015 and Hugonnier, Lester and Weill 2016 as exceptions).

Empirically, it is well known that inter-dealer markets have important asset pricing implications, while most earlier studies focus on equity markets (Hansch, Naik and Viswanathan, 1998, Reiss and Werner, 1998, Viswanathan and Wang, 2004, for example). Recently, several studies have documented the critical functions and potential instabilities of inter-dealer markets in various OTC market settings (Green, Hollifield and Schurhoff, 2006, Hollifield, Neklyudov and Spatt, 2016, Collin-Dufresne, Junge and Trolle, 2017, Di Maggio, Kermani and Song, 2017, Schultz, 2017) thanks to the increasing availability of previously inaccessible data. In the theoretical OTC mar-

ket literature, some papers feature an explicit while reduced-form (usually modeled as centralized) inter-dealer market to facilitate the analysis of other aspects; for example, Garleanu (2009), Lagos and Rocheteau (2009) and subsequent papers built on them consider unlimited asset holding positions by investors, and Babus and Parlatore (2018) consider how strategic traders choose a dealer to trade with. Our work complement them by characterizing what we believe are the most robust features of the OTC inter-dealer markets and isolating the impacts on asset liquidity.

2 The Model

Timeline, asset, and preferences. We fix a probability space satisfying the usual conditions. Time is continuous and runs infinitely: $t \in [0, \infty)$. Consider a market for a single indivisible consumption good. There are three types of agents: buyers, dealers, and sellers. All agents are risk-neutral with time preferences determined by a constant discount rate $r > 0$. The monetary values of a unit of the consumption good for a buyer, a dealer, and a seller are normalized at $\theta > 0$, 0, and 0, respectively. We also call buyers and sellers jointly as "customers" in what follows.

Buyers and sellers. At any time t, there are x_t active buyers and y_t active sellers present on the market, which will be endogenous determined. We denote by $z_t = x_t + y_t$ the total mass of customers. There are also n new buyers and n new sellers arriving at the market at any time t.[4] We focus on intermediated asset markets in the sense that buyers and sellers cannot trade with each other directly but only through dealers. A seller or a buyer who has successfully transacted with an dealer leaves the market. We also assume that at any time t, ηx_t active buyers and ηy_t active sellers exit the market without a successful transaction, where $\eta > 0$. This "exit shock" captures that a buyer or seller may lose her trading opportunity if cannot find an intermediary. For example, a junk bond mutual fund running out of cash has to find a dealer to sell its illiquid bonds, and it may default if cannot find a dealer timely. In this sense, the amount of active buyers and sellers also capture how many customers are actively waiting and searching for dealers in the economy.

Dealers. Also at any time t, there are μ_t active intermediaries participating in the market, out of $\bar{\mu} > 0$ total dealers. Dealers make participation and inventory holding decisions endogenously.

[4] The model can be extended to handle different amount of newly arrived buyers and sellers without changing the main economic mechanisms, but doing so significantly complicates the calculation.

Specifically, at any time t, an endogenous mass of μ_t dealers are active on the market, and an active dealer may hold 0, 1, or 2 units of the consumption good, with the inventory cost being 0, c, and ρc, respectively, which satisfy $c > 0$ and $\rho > 1$. The two parameters jointly summarize the inventory schedule of dealers: c captures the level of cost and ρ captures the curvature of the cost schedule.

We further make the following parametric assumption concerning the inventory cost schedule, suggesting that the marginal inventory cost is weakly increasing. This is economically relevant, but it is also weaker than the usual quadratic inventory cost assumption in the literature:

Assumption **1.** *Dealers' inventory cost schedule is weakly convex, that is,* $\rho \geqslant 2$.

We denote the endogenous masses of dealers who hold 0, 1, and 2 units of the good μ_t^0, μ_t^1, and μ_t^2, respectively, where $\mu_t^0 + \mu_t^1 + \mu_t^2 = \mu_t$. For the ease of exposition, we also call them type-0, type-1, and type-2 dealers, with the type indicating an dealer's inventory holding status which is endogenous.

It is worth noting that our setting purposefully relaxes the common assumption in the literature that limits the agents to asset holding positions, 0 or 1, in a particular way. By allowing a dealer to hold 0, 1, or 2 units of the good, we parsimoniously capture that a dealer may hold a high or low inventory in reality, which will ultimately give rise to a trading motive between a high-inventory-holding (i.e., type-2) dealer and a none-inventory-holding (i.e., type-0) dealer in equilibrium. We are also aware of the literature that allowing for unlimited or continuous level of asset holding positions.[5] As we will specify later, because we allow dealers to play mixed strategies in inventory holding decisions, our model effectively allows for a continuous level of asset holding positions.

Matching protocols. At any given point of time t, active agents search for counter-parties for a potential trade. To keep the model flexible, we assume that the matching protocols 1) between customers and dealers and 2) among dealers are two separate ones. These matching protocols can be micro-founded by the continuous time random matching framework in Duffie, Qiao and Sun (2017), which guarantees that the exact law of large numbers of Sun (2006) applies.

First, denote by $m(z_t, \mu_t)$ the primitive matching function between an active customer and an active dealer, which is weakly increasing in its both arguments. We assume that a customer meets

[5] Garleanu (2009) and Lagos and Rocheteau (2009) are among the first to allow agents to have unlimited asset holdings positions. Recently, Afonso and Lagos (2015) and Uslu (2017) further generalize this line of research without relying on a centralized inter-dealer market.

a dealer according to a Poisson process with rate α:

$$\alpha(z_t, \mu_t) = \frac{m(z_t, \mu_t)}{z_t},$$

which is decreasing in z_t. An alternative way to interpret α is that the probability of an active customer meeting an active dealer in a short time window $(t, t + dt]$ is αdt.

Similarly, an active dealer meets an active customer according to a Poisson process with rate β.

$$\beta(z_t, \mu_t) = \frac{m(z_t, \mu_t)}{\mu_t},$$

which is decreasing in μ_t. Intuitively, the property of these matching functions captures the competition among agents. Customers compete with each other in finding dealers while dealers also compete with each other in intermediating customer order flows.

Then, we assume that a dealer meets another dealer according to an independent Poisson process with rate λ, which is also independent to a dealer's inventory holdings. An alternative formulation is that the probability of an active dealer meeting another active dealer in a short time window $(t, t + dt]$ is λdt. Economically, λ parsimoniously captures the friction of the inter-dealer market: a larger λ suggests that intermediaries are easier to meet each other.

Bilateral trading. Trading prices are determined through reduced-form bargaining as standard in the literature (for example, Rubinstein and Wolinsky, 1987, Duffie, Garleanu and Pedersen, 2005). As a benchmark, we assume that all agents, when meet bilaterally, have equal bargaining powers.[6] Thus, an agent's bargaining position ultimately depends on the outside option, which in turn depends on the availability of other counter-parties both at time t and in the future. As a result, agents who meet always equally share the gains from any potential trade, provided the gains from trade are positive.

3 Equilibrium Analysis

As standard in the literature, in most of the paper we focus on stationary (i.e., steady-state) equilibria, that is, equilibria in which the mass of each type of agents is constant. Thus, we suppress

[6] The qualitative predictions of our model are robust to more sophisticated bargaining protocols that may give rise to unequal relative bargaining powers between the two bargaining parties.

the time argument in equilibrium variables. We denote by V_b, V_s, V_0, V_1, and V_2 the equilibrium value functions of an active buyer, seller, type-0 dealer, type-1 dealer, and type-2 dealer, respectively.

Since the game played by agents in our framework is essentially a dynamic bargaining game, it is clear that at any steady-state equilibrium, the sub-game played by two meeting agents j and k can be summarized by the following two-strategy game,

		Agent k	
		Accept	*Reject*
Agent j	*Accept*	$\frac{G(\{V_i\}_i)}{2}, \frac{G(\{V_i\}_i)}{2}$	$0, 0$
	Reject	$0, 0$	$0, 0$

where $G(\{V_i\}_i)$ denotes the potential gains from trade between the two meeting agents j and k, which are in turn determined endogenously by all the agents' value functions given the steady state of the dynamic game as well as agents' rational expectations of achieving the corresponding steady state. Intuitively, only when the two meeting agents both choose "Accept", trade will happen. In turn, only when the potential gains from trade are positive, the two meetings agents will choose "Accept" simultaneously. On the flip side, at least one agent will choose "Reject" when the potential gains from trade are negative, and thus a trade will not happen. When the potential gains from trade are zero, agents may play mixed strategies, where their equilibrium mixed strategies will be determined by the steady-state distribution of the mass of agents as well as their value functions.

Below we consider three types of equilibria, respectively, which exhaust all possible types of stationary pure-strategy equilibria. The first two types of equilibria represent the cases when dealers choose to participate in the market while the inter-dealer market may be either active or inactive. The last type of equilibrium represents a case where dealers do not participate and thus no trade happens. In the next section, we further characterize which equilibrium happens given model parameters and formulate dealers' strategic considerations behind their equilibrium behaviors.

To organize the analysis of different types of equilibria, we state two useful lemmas.

Lemma **1.** *In any steady-state equilibrium, the mass of active buyers x equals to that of active sellers y.*

Lemma 1 stems from the accounting identity that at any steady state, the inflow of goods

intermediated equals to the outflow of goods intermediated:

$$n - \eta x = n - \eta y \,, \tag{3.1}$$

implying $x = y$. We highlight that this is not an assumption but instead an equilibrium outcome.

LEMMA **2.** *In any steady-state equilibrium with $\mu > 0$, inter-dealer trade happens only if $\mu_2 > 0$.*

Economically, Lemma 2 amounts to saying that only when the dispersion of dealers' inventory holdings is large enough, that is, only when there are some dealers holding a relatively high level of inventory in equilibrium. In other words, $\mu_2 > 0$ is a necessary condition for the inter-dealer market being active. Intuitively, if there were only type-0 and type-1 dealers in the market, no inter-dealer trade would happen because of the lack of gains from trade. Instead, a type-2 dealer may potentially trade with a type-0 dealer and both become type-1 dealers, because the gains from such a trade may be positive. As a result, we will use whether μ_2 is positive in equilibrium to organize the discussion of the first two types of equilibria. Importantly, we will further show that inter-dealer trading must happen when a type-2 dealer meets a type-0 dealer. In other words, $\mu_2 > 0$ is also a sufficient condition for the inter-dealer market being active.

3.1 Equilibrium with inter-dealer market: $\mu > 0$ and $\mu_2 > 0$

We first characterize a type of equilibria where $\mu > 0$ and $\mu_2 > 0$. We characterize the equilibrium conditions and show that the inter-dealer market is active in the sense that when a type-2 dealer meets a type-0 dealer, a transaction must happen and thus both become type-1 dealers. Figure 1 below illustrates this type of equilibria, where solid arrows indicate the flows of goods and dashed arrows indicate the flows of customers in and out of the market as well as dealer's changes in inventory-holding status.

In an equilibrium, the order flows to the buyers must equal the order flows from the dealers:

$$n - \eta x = x\alpha \frac{\mu_1 + \mu_2}{\mu} \,, \tag{3.2}$$

and similarly, the order flows from the sellers must equal the order flows taken by the dealers:

$$n - \eta y = y\alpha \frac{\mu_0 + \mu_1}{\mu} \,. \tag{3.3}$$

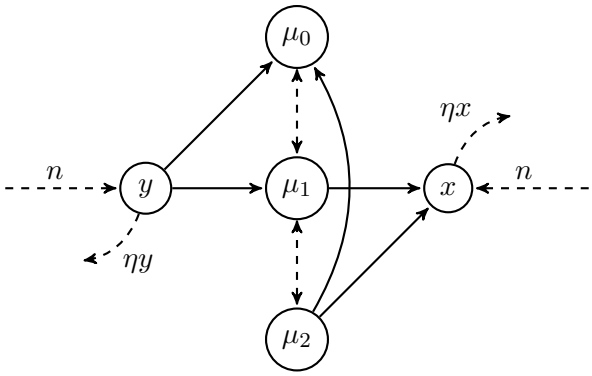

Figure 1: Equilibrium with the inter-dealer market

Together with Lemma 1, these two accounting identities imply that

$$\mu_0 = \mu_2 \,, \tag{3.4}$$

that is, the mass of type-0 dealers must equal that of type-2 dealers.

For convenience, define

$$\mu_I = \mu_0 = \mu_2 \,.$$

Intuitively, a potential inter-dealer trade can only happen between these two types of dealers. Notice that these two dealers become type-1 dealers after such an inter-dealer trade, suggesting that the equilibrium mass of type-1 dealers cannot be too small. In other words, the equilibrium mass of either type-0 or type-2 dealers cannot be too large. The following result shows that the share of type-0 or type-2 dealers indeed has an upper bound, guarantees the potential for an inter-dealer trade.

Lemma **3.** *In any equilibrium with $\mu > 0$ and $\mu_2 > 0$, there must be that $0 < \frac{\mu_I}{\mu} \leqslant \frac{1}{3}$. In other words, $\frac{\mu_1}{\mu} \geqslant \frac{1}{3}$.*

We can now derive the agents' equilibrium Hamilton-Jacobi-Bellman (HJB) equations, taking

into account the endogenous matching and bargaining outcomes:

$$
\begin{aligned}
(\eta + r)V_b &= \alpha \left[\frac{\mu_2}{\mu} \frac{1}{2}(\theta + V_1 - V_2 - V_b) + \frac{\mu_1}{\mu} \frac{1}{2}(\theta + V_0 - V_1 - V_b) \right], && (3.5) \\
(\eta + r)V_s &= \alpha \left[\frac{\mu_0}{\mu} \frac{1}{2}(V_1 - V_0 - V_s) + \frac{\mu_1}{\mu} \frac{1}{2}(V_2 - V_1 - V_s) \right], && (3.6) \\
rV_0 &= \beta \frac{y}{x+y} \frac{1}{2}(V_1 - V_0 - V_s) + \lambda \frac{\mu_2}{\mu} \frac{1}{2}(2V_1 - V_0 - V_2), && (3.7) \\
rV_1 &= \beta \left[\frac{x}{x+y} \frac{1}{2}(\theta + V_0 - V_1 - V_b) + \frac{y}{x+y} \frac{1}{2}(V_2 - V_1 - V_s) \right] - c, && (3.8) \\
rV_2 &= \beta \frac{x}{x+y} \frac{1}{2}(\theta + V_1 - V_2 - V_b) + \lambda \frac{\mu_0}{\mu} \frac{1}{2}(2V_1 - V_0 - V_2) - \rho c. && (3.9)
\end{aligned}
$$

These HJB equations are intuitive. First, since all agents have equal bargaining power, they always equally share the potential gains from trade, if positive. The bargaining process also determines the trading prices in every bilateral trading.

Second, a buyer may buy a unit of good either from a type-2 or a type-1 dealers, while a seller may sell to either a type-0 or type 2 dealer. This is reflected in the buyer's and seller's HJB equations (3.5) and (3.6).

Finally, for the three types of dealers, they all have their unique pattern of trading, reflected by (3.7), (3.8), and (3.9). A type-0 dealer can either buy a unit of good from a seller or from a type-2 dealer, a type-1 dealer can either buy from a seller or sell to a buyer, while a type-2 dealer can either sell to a buyer or to a type-0 dealer. Consistent with our earlier argument, only the type-0 and type-2 dealers are potential candidates for an inter-dealer trade.

Analyzing these value functions, we first show that an inter-dealer trade must happen as long as a type-0 dealer meets a type-2 dealer.

Lemma **4.** *In any equilibrium with $\mu > 0$ and $\mu_2 > 0$, $2V_1 - V_0 - V_2 > 0$ when Assumption 1 holds. That is, the gains from trade from a potential trade between a type-0 and a type-2 dealer is always strictly positive and thus the inter-dealer market is active.*

In illustrating the economic intuition behind Lemma 4, it is worth noting that Assumption 1 about the inventory cost schedule being weakly convex is one (weak and economically relevant) sufficient condition for the inter-dealer market being active, but not a necessary condition. In the proof for Lemma 4, we show that the gains from trade from a potential trade between a type-0 and

a type-2 dealer can be expressed as

$$2V_1 - V_0 - V_2 = \frac{(\rho - 2)c + \kappa_1 \beta}{\kappa_2},$$

where $\kappa_1 > 0$ and $\kappa_2 > 0$ are two strictly positive constants that are determined in equilibrium. It shows that the potential gains from trade are captured by two independent terms. The first term captures an instantaneous effect: it shows that an inter-dealer trade allows the two dealers to jointly save their inventory costs, consistent with the classical view of the inventory sharing. The second term captures a continuation effect: it implies that an inter-dealer trade allows the two dealers, who subsequently become two type-1 dealers, to jointly handle more order flows from customers. Since β is always strictly positive in any equilibrium, an inter-dealer trade happens as long as the second effect dominates, even if Assumption 1 is not satisfied.

So far, we have shown that $\mu_2 > 0$ is both a sufficient and a necessary condition for the inter-dealer market being active. This verifies our equilibrium categorization such that we call an equilibrium where $\mu > 0$ an equilibrium with an inter-dealer market while that where $\mu = 0$ an equilibrium without an inter-dealer market.

A natural and important question is when an equilibrium with an inter-dealer market exists. In principle, the existence of such an equilibrium requires all the value functions V_i, $i \in \{b, s, 0, 1, 2\}$ to be weakly positive and all the gains from trade from the five possible trades as illustrated by the five solid arrows in Figure 1 as well as in the right hand sides of value functions (3.5), (3.6), (3.7), (3.8), and (3.9) to be weakly positive. It is not surprising, however, that many of these ten constraints will be slack in equilibrium. In the following theorem, we show the single key condition that guarantees the simultaneously satisfaction of the ten constraints:

THEOREM **1.** *For any $\mu > 0$, an equilibrium with an inter-dealer market exists if*

$$V_2 - V_1 - V_s \geqslant 0, \tag{3.10}$$

where V_s, V_1 and V_2 satisfy the value functions (3.5), (3.6), (3.7), (3.8), and (3.9).

Theorem 1 implies that the only criterion for the emergence of the inter-dealer market is that a type-1 dealer is willing to trade with a seller, given the corresponding mass distribution of agents

in the corresponding equilibrium. Intuitively, this criterion is critical because only when a type-1 dealer is willing to take a seller's order into its balance sheet and effectively increases its inventory holding, a positive mass of type-2 dealers can exist in equilibrium, which by our earlier argument justifies the emergence of the inter-dealer market.

Following Theorem 1, we may further formulate the equilibrium criterion (3.10) with respect to the model parameters which capture the economic environment. The following proposition gives results along several economically relevant dimensions:

PROPOSITION **1.** *For any $\mu > 0$:*

i). there exists a lower threshold $\underline{\theta}$ such that for all $\theta \geqslant \underline{\theta}$, an equilibrium with $\mu_2 > 0$ exists;

ii). there exists a upper threshold $\overline{c}$ such that for all $c \leqslant \overline{c}$, an equilibrium with $\mu_2 > 0$ exists;

iii). there exists a upper threshold $\overline{\rho}$ such that for all $\rho \leqslant \overline{\rho}$, an equilibrium with $\mu_2 > 0$ exists.

Proposition 1 implies that the inter-dealer market will endogenously emerge 1) when the valuation between the buyer and the seller is different enough, that is, the potential gains from trade by intermediating the good from the seller to the buyer are large enough, 2) when dealers' inventory cost is lower enough, or 3) when dealers' inventory cost schedule is not too convex. Overall, when dealers' inventory cost is not too high relative to the potential gains from trade by intermediating the good, dealers are more likely to hold a high level of inventory in equilibrium and accordingly trade with each other to better manage their inventory holdings.

3.2 Equilibrium without inter-dealer market: $\mu > 0$ and $\mu_2 = 0$

Next, we characterize a type of equilibria where $\mu > 0$ and $\mu_2 = 0$. Figure 2 below illustrates this type of equilibria, where again solid arrows indicate the flows of goods and dashed arrows indicate the flows of customers in and out of the market as well as dealer's changes in inventory-holding status.

With the help of Lemmas 2 and 4, we have already known that $\mu_2 > 0$ is a sufficient and necessary condition for the inter-dealer market being active. Equivalently, this means that $\mu_2 = 0$ is also a sufficient and necessary condition for the inter-dealer market being inactive.

Similarly, we can derive the agents' equilibrium Hamilton-Jacobi-Bellman (HJB) equations,

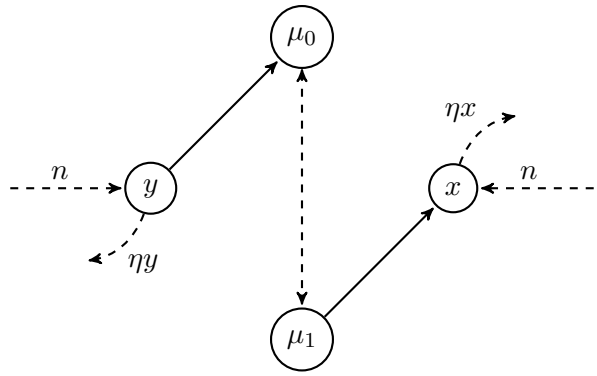

Figure 2: Equilibrium without the inter-dealer market

taking into account the endogenous matching and bargaining outcomes:

$$(\eta + r)V_b = \alpha \frac{\mu_1}{\mu} \frac{1}{2}(\theta + V_0 - V_1 - V_b)\,, \tag{3.11}$$

$$(\eta + r)V_s = \alpha \frac{\mu_0}{\mu} \frac{1}{2}(V_1 - V_0 - V_s)\,, \tag{3.12}$$

$$rV_0 = \beta \frac{y}{x+y} \frac{1}{2}(V_1 - V_0 - V_s)\,, \tag{3.13}$$

$$rV_1 = \beta \frac{x}{x+y} \frac{1}{2}(\theta + V_0 - V_1 - V_b) - c\,. \tag{3.14}$$

Also similarly, to support such an equilibrium, we need to verify that all the four value functions V_i, $i \in \{b, s, 0, 1\}$ are weakly positive and both the gains from trade from the two possible trades as illustrated by the two solid arrows in Figure 2 are weakly positive. Importantly, to support an equilibrium with $\mu_2 = 0$, we need to further verify that a type-1 dealer would not find it profitable if it were to buy from an active seller. In other words, a type-1 dealer would never deviate from such an equilibrium by taking an inflow order from an active seller. Suppose, an individual type-1 dealer deviates and becomes a type-2 dealer. Its hypothetical value function satisfies the following HJB equation:

$$rV_2 = \beta \frac{x}{x+y} \frac{1}{2}(\theta + V_1 - V_2 - V_b) + \lambda \frac{\mu_0}{\mu} \frac{1}{2}(2V_1 - V_0 - V_2) - \rho c\,, \tag{3.15}$$

and the existence of an equilibrium with $\mu_2 = 0$ requires that

$$V_2 - V_1 - V_s \leqslant 0\,. \tag{3.16}$$

In the following theorem, we show the single key condition that guarantees the simultaneously satisfaction of the seven constraints:

THEOREM **2.** *For any $\mu > 0$, an equilibrium without an inter-dealer market exists if*

$$V_2 - V_1 - V_s \leqslant 0\,, \tag{3.17}$$

where V_s, V_1 and V_2 satisfy the value functions (3.11), (3.12), (3.13), (3.14), and (3.15).

Parallel to Theorem 1, Theorem 2 implies that the only criterion for the closure of the inter-dealer market is that a type-1 dealer is not willing to trade with a seller, given the corresponding mass distribution of agents in the corresponding equilibrium.

It is important to note that the value functions V_s, V_1 and V_2 in condition (3.17) are different from those in condition (3.10). This is because the value functions are calculated differently using different HJB equations, which in turn depend on the specific trading patterns and distributions of agents in the two types of equilibria. As a result, mathematically, condition (3.17) is almost surely not the flip side of condition (3.17). The satisfaction of condition (3.17) does not necessarily suggest that condition (3.10) is violated, and vice versa.

However, the economic coincidence of conditions (3.10) and (3.17) is important. It suggests that whether a type-1 dealer is willing to trade with a seller is an economically critical condition to determine which type of equilibrium ultimately emerges, taking other agents' decisions as given. This point will be illustrated more forcefully when we analyze the strategic considerations of dealers' inventory holding decisions later.

Also, we may formulate the equilibrium criterion (3.17) with respect to the model parameters which capture the economic environment:

PROPOSITION **2.** *For any $\mu > 0$:*

i). there exists a upper threshold $\overline{\theta}$ such that for all $\theta \leqslant \overline{\theta}$, an equilibrium with $\mu_2 = 0$ exists;

ii). there exists a lower threshold $\underline{c}$ such that for all $c \geqslant \underline{c}$, an equilibrium with $\mu_2 = 0$ exists;

iii). there exists a lower threshold $\underline{\rho}$ such that for all $\rho \geqslant \underline{\rho}$, an equilibrium with $\mu_2 = 0$ exists.

Proposition 2 implies that the inter-dealer market will not emerge 1) when the valuation between the buyer and the seller is too close, that is, the potential gains from trade by intermediating the

good from the seller to the buyer are small enough, 2) when dealers' inventory cost is high enough, or 3) when dealers' inventory cost schedule is too convex.

3.3 No-trade equilibrium: $\mu = 0$

Since dealers can endogenously choose to participate in the market or not, there may exist a third type of pure-strategy equilibria where none of the dealers are active, that is, $\mu = 0$. In such an equilibrium, there must be $V_b = V_s = 0$, and any dealer would not find it profitable if it were to participate. Suppose a dealer deviates and participates in the market. Its hypothetical value functions would satisfy:

$$rV_0 = \beta_0 \frac{y}{x+y} \frac{1}{2}(V_1 - V_0)\,, \tag{3.18}$$

$$rV_1 = \beta_0 \left[\frac{x}{x+y} \frac{1}{2}(\theta + V_0 - V_1) + \frac{y}{x+y} \frac{1}{2}(V_2 - V_1) \right] - c\,, \tag{3.19}$$

$$rV_2 = \beta_0 \frac{x}{x+y} \frac{1}{2}(\theta + V_1 - V_2) - \rho c\,, \tag{3.20}$$

where β_0 is defined by

$$\beta_0 = \lim_{\mu \to 0} \frac{m(z,\mu)}{\mu} \leqslant +\infty\,.$$

Intuitively, no dealer is willing to participate in the market when $V_0 \leqslant 0$. From the above conditions (3.18), (3.20) and (3.20), we can show that V_0 is a linear and increasing function of θ. Intuitively, if θ is too small and β_0 is finite, a dealer will not find it profitable to participate in the market even if it can capture all the potential customer flows.

We have the following formal result:

Proposition **3.** *There exists a threshold θ_0 such that for all $\theta < \theta_0$, an equilibrium with $\mu = 0$ exists.*

Notably, Proposition 3 suggests that dealers may not find it optimal to participate in the market even if the physical cost of participation is zero. This is still intuitive, because participation in equilibrium means that a dealer must hold some non-zero inventory on average. This is consistent with the idea that a dealer is economically defined by that it holds inventories; otherwise the dealer essentially becomes a broker.

4 Strategic Interaction of Intermediation

Having characterized all the possible types of equilibria, we consider which equilibrium endogenously happens given the economic environment.

So far, Propositions 1 and 2 suggest that an inter-dealer market will endogenously emerge when dealers' inventory cost schedule is sufficiently low, and vice versa. But as suggested by Theorems 1 and 2, the more interesting cases are that when dealers' inventory cost is in some intermediate region, the criteria (3.10) and (3.17) may both hold, or are both violated. In other words, the equilibria with and without an inter-dealer market may co-exist, or none of them exists. We formulate these situations below. In doing so, we illustrate the coordination motives behind the dealers' inventory holding decisions, which stem from the potential for forming the inter-dealer market.

First, recall that Theorems 1 and 2 suggest that whether a type-1 dealer is willing to take an order from an active seller and to effectively increase its inventory is the solely important criterion to determine which type of equilibria is sustainable, given the equilibrium distribution and values of other agents in the corresponding equilibrium. To formulate type-1 dealers' strategy as well as their willingness to trade requires us to analyze the bargaining (sub-)game between an type-1 dealer and an active seller, when they meet each other:

		Seller	
		Accept	*Reject*
Type-1 Dealer	*Accept*	$\frac{V_2-V_1-V_s}{2}, \frac{V_2-V_1-V_s}{2}$	$0, 0$
	Reject	$0, 0$	$0, 0$

As suggested by the bargaining (sub-)game above, the type-1 dealer's inventory decision of whether or not to increase its inventory holding by taking an order from a seller is solely determined by whether $V_2 - V_1 - V_s$ is positive or negative, given the endogenously determined value functions in the corresponding equilibrium. Formally, we define the notion of strategic complementarity as follows:

Definition **1.** *For any given set of model parameters, an dealer's inventory holding decision exhibits strategic complementarity if $V_2 - V_1 - V_s$ is larger in a steady-state outcome where all the other dealers and sellers choose "Accept" than in a comparable steady-state outcome where all the*

other dealers choose "Reject".

Definition 1 has the following intuitive interpretation given that a seller always plays "Accept". If all the other dealers play "Accept", it must be that $\mu_2 > 0$ and the value functions will be governed by (3.5), (3.6), (3.7), (3.8), and (3.9). In this case, an inter-dealer market emerges. On the flip side, if all the other dealers play "Reject", it must be that $\mu_2 = 0$ and the value functions will be governed by (3.11), (3.12), (3.13), (3.14), and (3.15). In this case, an inter-dealer market will not emerge. Thus, Definition 1 amounts to saying that if a type-1 dealer's payoff gain of increasing its inventory (and thus effectively participating in the inter-dealer market) is higher when other type-1 dealers also do so (and thus the inter-dealer market emerges), then type-1 dealers' inventory holding decision exhibits strategic complementarity. This argument also makes it clear that such strategic complementarity indeed comes from the potential for forming the inter-dealer market.

To handle the case where $V_2 - V_1 - V_s = 0$, we further allow the agents to play mixed strategies. Specifically, a type-1 dealer's strategy in the above bargaining (sub-)game is $(p, 1-p)$ while a seller's strategy is $(q, 1-q)$, whenever they meet each other. In this case, a trade between a type-1 dealer and a seller happens with probability pq when they meet. Notice that the bargaining (sub-)game itself is not sufficient to determine the equilibrium profile (p, q). Rather, the equilibrium probability pq at which a trade between a type-1 dealer and a seller happens will be determined by the condition $V_2 - V_1 - V_s = 0$ as well as other value functions as prescribed by (3.5), (3.6), (3.7), (3.8), and (3.9).

4.1 Equilibrium multiplicity

With Definition 1, we formally characterize under what conditions multiple equilibria may emerge, that is, the equilibria with and without an inter-dealer market may co-exist.

THEOREM **3.** *For any given set of model parameters, if $V_2 - V_1 - V_s$ is positive in a steady-state outcome where all the other dealers and sellers choose "Accept" and is negative in a comparable steady-state outcome where all the other dealers choose "Reject", then an equilibrium with $\mu = \bar{\mu}$, $\mu_2 > 0$ and an equilibrium with $\mu = \bar{\mu}$, $\mu_2 = 0$ co-exist.*

Theorem 3 is intuitive. The condition underlying Theorem 3 is a stronger form of the notion of strategic complementarity as defined in Definition 1. It implies that a type-1 dealer finds it optimal

to take the inflow order from an active seller when all the other type-1 dealers also choose to take orders from sellers, while the same type-1 dealer finds it optimal not to trade with a seller when all the other type-1 dealers also refrain from trading with active sellers. As a result, whether a type-1 dealer trades with an active seller and becomes a type-2 dealer depends on its belief about other dealers' strategies. This resembles the classic notion of coordination in complete information static games and eventually gives rise to multiple steady-state equilibria in our dynamic framework.

We provide an numerical example to illustrate Theorem 3. In this example, we choose $\bar{\mu} = 1$, $c = 0.2$, $\rho = 2$, $n = 1$, $\eta = 1$, $r = 1$, $\lambda = 0.05$, and $m(z, \mu) = 5 \min\{z, \mu\}$.

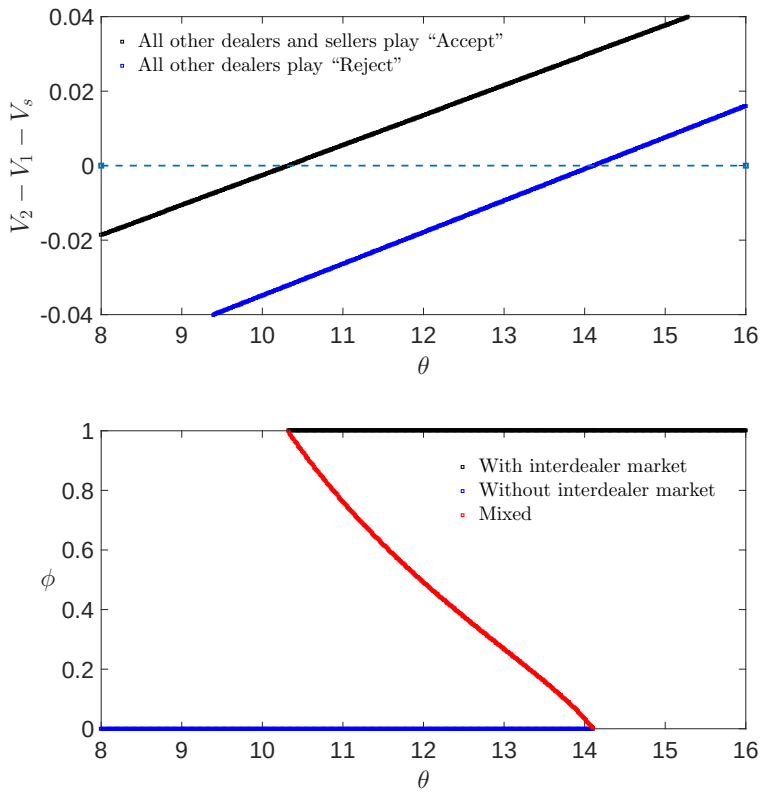

Figure 3: The correspondence of payoff gains and equilibrium multiplicity

The top panel of Figure 3 plots $V_2 - V_1 - V_s$, the payoff gains of a type-1 dealer trading with an active seller given other type-1 dealers' strategy, against different values of θ. It shows that in the economic environment captured by the given parameters, a type-1 dealer's inventory holding strategy exhibits strategic complementarity for any θ. In particular, for intermediate levels of θ, $V_2 - V_1 - V_s$ is positive in a steady-state outcome where all the other dealers and sellers choose "Accept" and is negative in a comparable steady-state outcome where all the other dealers choose

"Reject". Therefore, Theorem 3 holds, suggests equilibrium multiplicity within that intermediate range of θ. This is indeed confirmed by the bottom panel of Figure 3, which plots the emergence of the inter-dealer market given different values of θ.

Intuitively, when θ is sufficiently large (small), the inter-dealer market endogenously emerges (closes). When θ is intermediate, whether the inter-dealer market will emerges depends on a type-1 dealer's belief about other dealers' beliefs of forming the inter-dealer market. In the bottom panel of Figure 3, the downward-sloping part of the reserve-Z curve illustrates another unstable mix-strategy equilibrium when θ is within this intermediate region, again consistent with the equilibrium predictions of classic complete information coordination games.

4.2 Liquidity implications

A natural question is how much liquidity the dealers provide to the customers. With Lemma 1 and the underlying intuition that all realized good flows between buyers and sellers must be intermediated by the dealers, we formally define the notion of liquidity in our economy as follows.

Definition **2.** *The equilibrium liquidity of the asset market is defined by the good intermediated per unit of time:*

$$L = n - \eta x \,. \tag{4.21}$$

On the flip side, this definition implies that we can also use the mass of waiting customers $2x = x + y$ to measure asset illiquidity.

The following formal result suggests that trading among intermediaries unambiguously improve liquidity:

Proposition **4.** *When multiplicity happens, asset liquidity is always higher in an equilibrium with $\mu_2 > 0$ than the comparable equilibrium with $\mu_2 = 0$.*

Proposition 4 is important because it suggests that an equilibrium with the inter-dealer market is unambiguously better than a comparable equilibrium without the inter-dealer market in terms of liquidity provision. This allows us to formally rank the two equilibria over the dimension of asset market liquidity, which ultimately allows us to speak to the implications on market fragility, as we formulate below.

5 Market Fragility

A natural question is how does market fragility relate to the friction of the inter-dealer market: when it becomes easier for the intermediaries to meet each other, would it become more or less likely for multiple equilibria to occur?

For this purpose, we explicitly separate fundamental characteristics of our economy, including potential gains from trade θ and inventory costs c and ρ, from non-fundamental characteristics, in particular, the meeting rate of the inter-dealer market λ. We formally define market fragility as follows.

Definition **3.** *For a family of economies parameterized by θ, c and ρ, it is fragile if there exists a non-zero subset $\{\theta, c, \rho\}$ of the space $\mathbb{R}_+^2 \times (1, +\infty)$ in which multiple equilibria occur.*

Economically, the definition captures the idea that if market liquidity may suddenly jump even when fundamental varies continuously, the market is fragile.

To facilitate further solving the model, from now we focus on the fundamental parameter regions in which $\mu > 0$. We take the primitive matching function to be Leontief:

$$m(z_t, u_t) = k \min(z_t, u_t)\,,$$

and we also impose the following parametric restriction to ensure that an equilibrium with inter-dealer market exists under the Leontief matching function:

$$\frac{3n}{\mu} > \left(\frac{2k}{3} + \eta\right) \frac{r}{\eta + r}\,.$$

5.1 A frictionless benchmark

We first consider a benchmark under which dealers can meet each other frictionlessly. Concretely, we consider the possibility of market fragility when $\lambda \to \infty$. This resembles the modeling of a centralized inter-dealer market in the literature, for example, Garleanu (2009) and Lagos and Rocheteau (2009). The following proposition suggests that market fragility never happens under this frictionless benchmark.

Proposition **5.** *For any family of economies parameterized by θ, c and ρ, these exists a threshold*

$\bar{\lambda}$ such that the family of economies is not fragile when $\lambda > \bar{\lambda}$. Specifically, only the equilibrium with inter-dealer market $\mu_2 > 0$ happens.

Proposition 5 suggests that the equilibrium with an inter-dealer market is the unique equilibrium regardless of the fundamentals θ, c and ρ, as long as the friction of the inter-dealer market is sufficiently small. Intuitively, a more efficient inter-dealer market implies that it is easier for dealers to offload their inventory holdings. This makes every dealer more willing to hold more inventory regardless of other dealers' strategy, eventually yielding a unique equilibrium.

5.2 Market friction and market fragility

We then explore the case when the inter-dealer market friction becomes large. To help illustrate the mechanism, we consider what happens when it is extremely hard for dealers to meet each other, that is, λ is close to 0. The following proposition characterizes when market fragility happens.

PROPOSITION **6.** *For any family of economies parameterized by θ, c and ρ, these exists a threshold $\underline{\lambda}$ such that the family of economies is fragile when $\lambda < \underline{\lambda}$.*

The economic intuition underlying Proposition 6 is the coordination motives in dealers' inventory holding and liquidity provision decisions. When the inter-dealer market is less efficient, holding more inventory becomes more costly. Dealers' inventory holding decisions thus exhibit strategic complementarity: when other dealers hold more inventory and intermediate more order flows, less order flows go to the dealer in question, dampening its ability to make profits. Thus, the dealer in question also become more willing to hold more inventory despite the effectively higher cost. This strategic complementarity is stronger when the inter-dealer market is less efficient, leading to market fragility.

6 Comparative statics

To provide more visual guidance and intuition regarding market fragility, we provide several numerical comparative statics to illustrate under what conditions market fragility is more likely to occur, and what are the impact on market liquidity. To focus on trading among intermediaries, we restrict our attention to the parameter region where dealers are willing to participate under any equilibrium. We fix the following common parameters: $\bar{\mu} = 1$, $n = 1$, and $r = 1$.

6.1 Inter-dealer market friction and market fragility

One of the key ideas of our model is that equilibrium multiplicity may arise due to the coordination motives of dealers in inventory holding and liquidity provision decisions. The coordination motives are in turn affected by the friction of the inter-dealer market: when dealers are easier to meet and thus trade with each other, one dealer's inventory holding and liquidity provision decisions depend less on other dealers', thereby weaken the strategic complementarity and eventually reduces potential market fragility. The following Figure 4 illustrates this idea, using the inventory cost concavity parameter $\rho = 2.5$ and search friction parameter $k = 5$ under the Leontief matching function, that is, $m(z, \mu) = 5 \min\{z, \mu\}$, and exit shock parameter $\eta = 1$.

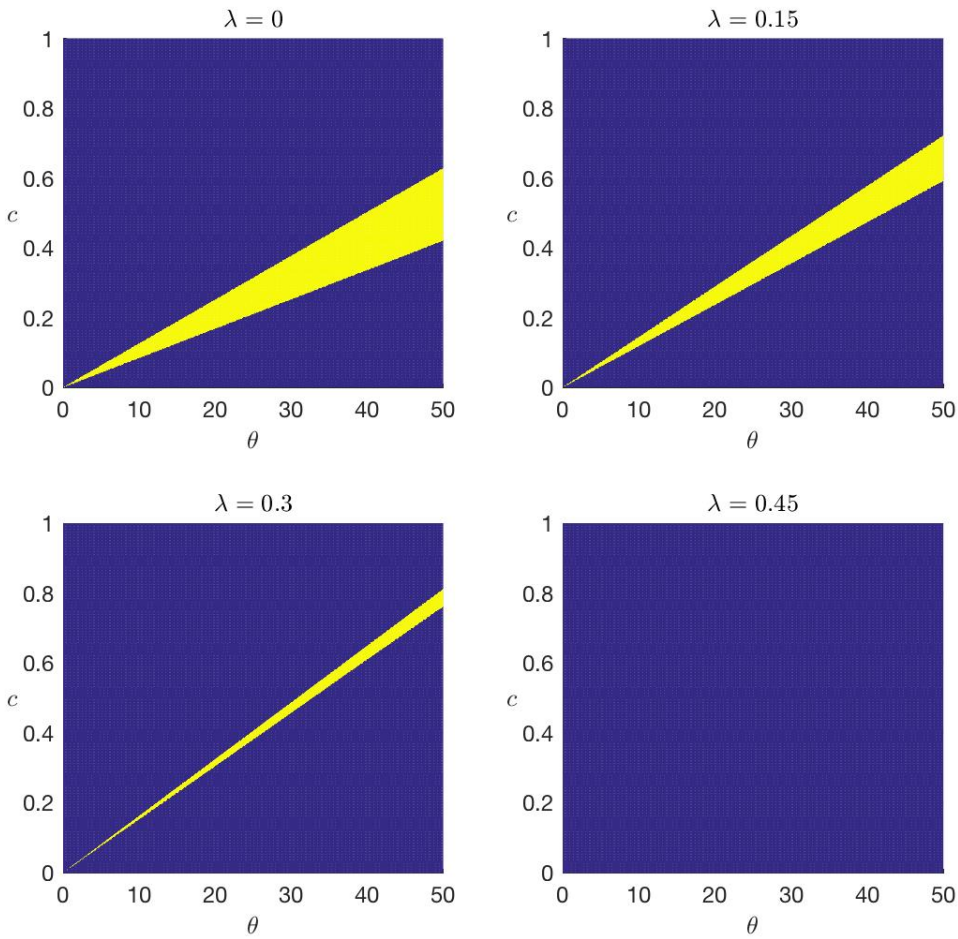

Figure 4: Fragility: yellow indicate multiplicity in the $(\theta, c)-$ parameter space

The four panels in Figure 4 show that, as the friction of the inter-dealer market becomes less severe, captured by a reduced λ, the area where multiplicity happens over the $(\theta, c)-$region becomes

smaller. Specifically, as λ takes values from 0 to 0.15 and then to 0.3, the family of economics is always fragile, but becomes less fragile in the sense that multiplicity is less likely to happen. When the inter-dealer market becomes sufficiently efficient as $\lambda = 0.45$, the strategy complementarity among dealer' inventory holding decisions is completely offset, and thus multiplicity cannot happen for any admissible parameters.

There are two important take-aways from Figure 4. First, the possibility of market fragility, measured by the area in the $(\theta, c)-$region where multiplicity happens, strictly decreases as the friction of the inter-dealer market becomes smaller. This is consistent with the theoretical predictions of Propositions 5 and 6.

Second, when the family of economies is fragile for a given level of inter-dealer market friction, equilibrium multiplicity always happens when the ratio of the potential gains from trade to the inventory cost lies in an intermediate range. Only in such cases, dealers' inventory holding strategy exhibits strategic complementarity. This is consistent with the theoretical predictions of Propositions 1 and 2.

6.2 Inventory cost, market fragility, and liquidity

We also explore the impacts of inventory costs on market fragility and liquidity across different parameter regions. The results are illustrated in Figures 5 and 6 below.

First, Figure 5 show the results when the inter-dealer market friction λ and the level of inventory cost c vary, under the same inventory cost concavity $\rho = 2.5$, search friction parameter $k = 5$, and exit shock friction $\eta = 1$. Specifically, the left three panels in Figure 5 features a family of economics where fragility happens. Here, the inter-dealer market is extremely inefficient in the sense that dealers cannot meet with each other at all (i.e., $\lambda = 0$). As the level of inventory cost c becomes higher from 0.1 to 0.2 and then to 0.4, the equilibrium without an active inter-dealer market is more likely to happen. In particular, multiplicity is more likely to happen as the inventory cost becomes higher. Interestingly, when the inventory cost is higher, multiplicity is more likely to happen when the potential gain from trade per unit of good is higher.

In contrast, the right three panels in Figure 5 features a family of economics where fragility does not happen. Here, the inter-dealer market is efficient in the sense that dealers can meet with each other at a sufficiently large rate (i.e., $\lambda = 0.4$). As the inventory cost becomes higher from

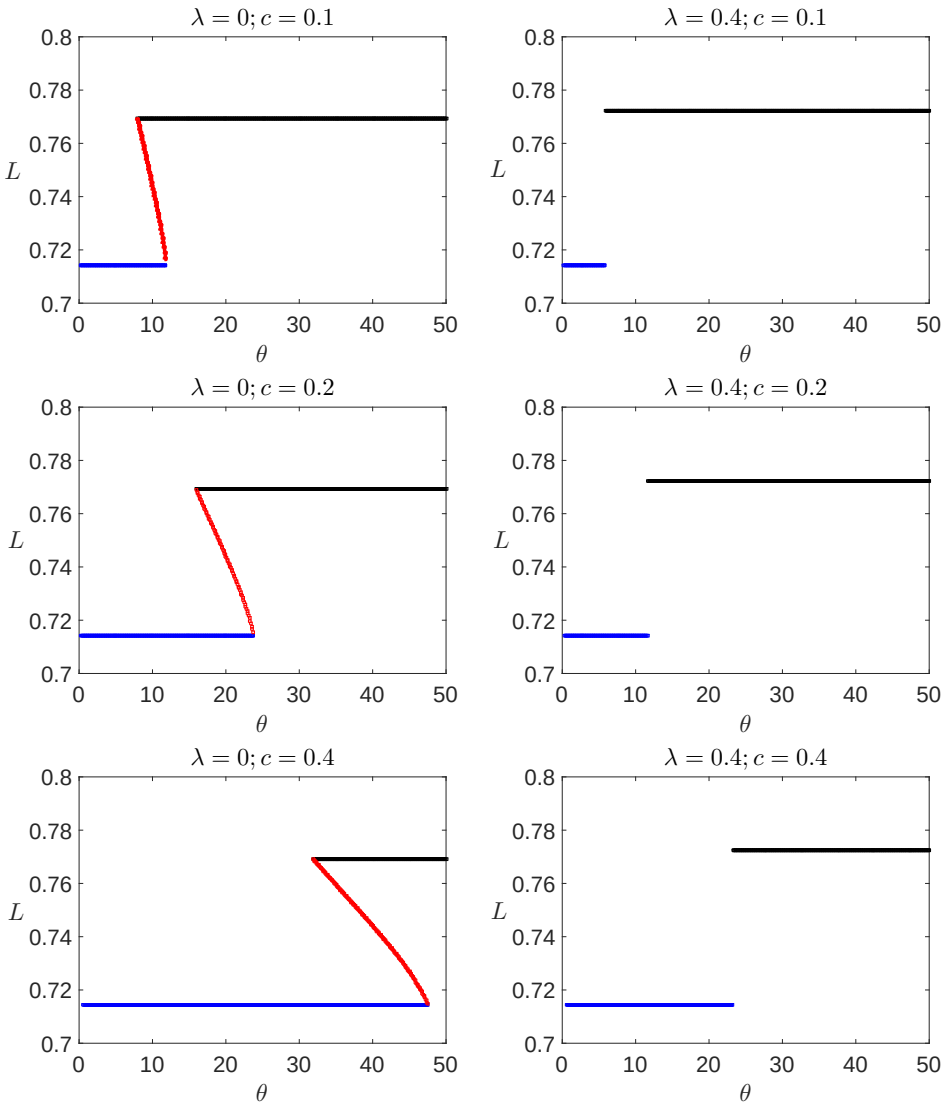

Figure 5: Equilibrium liquidity and fragility when λ and c vary

0.1 to 0.2 and then to 0.4, the equilibrium without an active inter-dealer market is more likely to happen. However, multiplicity never happens. These results are consistent with the theoretical predictions of Propositions 5 and 6.

In a similar fashion, Figure 6 shows the results when the inter-dealer market friction λ and the inventory cost concavity ρ vary, under the same level of inventory cost $c = 0.1$ and search friction $\eta = 1$. As the inventory cost concavity parameter ρ becomes higher from 2 to 4 and then to 6, the equilibrium without an active inter-dealer market is more likely to happen, but market fragility happens only when the inter-dealer market friction is higher (i.e., $\lambda = 0$). In particular, multiplicity is more likely to happen as the inventory cost schedule becomes more concave. These results are

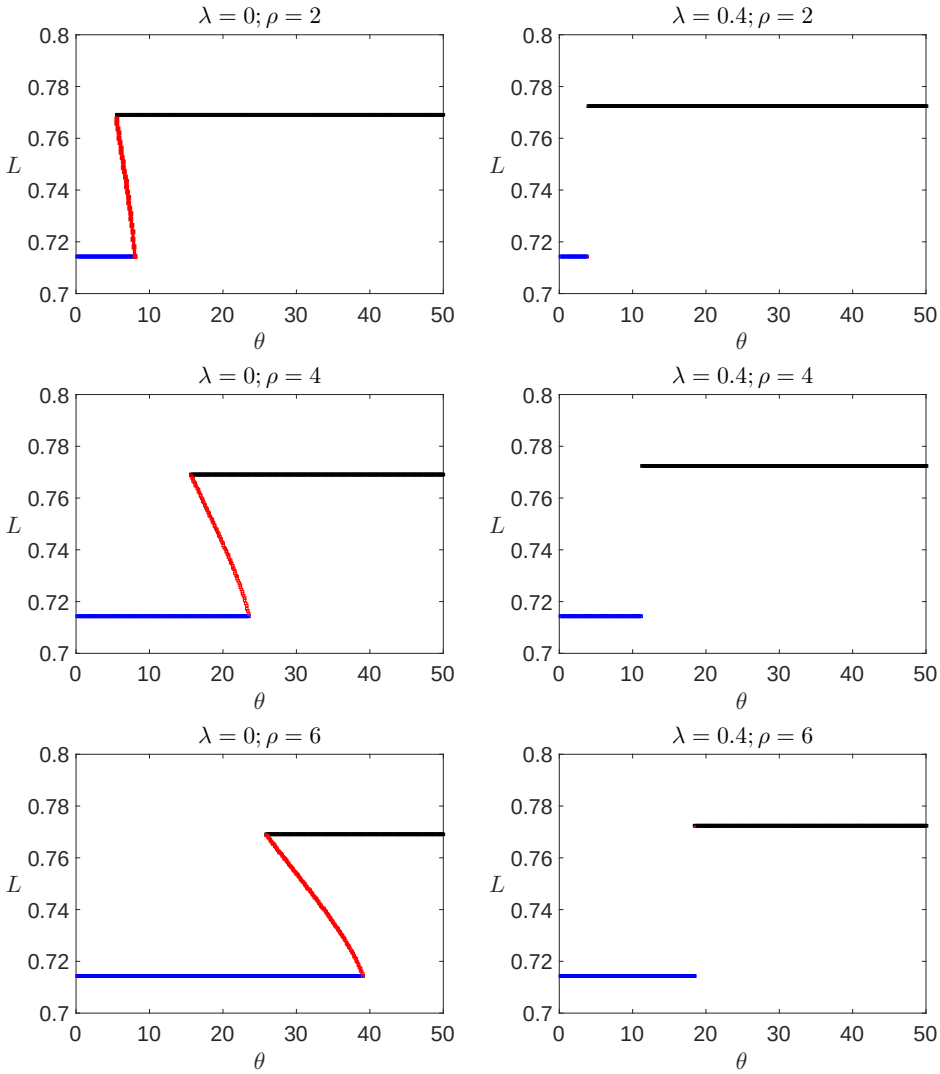

Figure 6: Equilibrium liquidity and fragility when λ and ρ vary

again consistent with the theoretical predictions of Propositions 5 and 6.

6.3 Search friction, market fragility, and liquidity

We next explore the impacts of search friction between customers and dealers on market fragility and liquidity across different parameter regions. The results are illustrated in Figure 7.

Here, we consider varying the inter-dealer market friction λ and the search friction parameter k between customers and dealers, under the same inventory schedule $c = 0.1$ and $\rho = 2.5$, and the exit shock parameter $\eta = 1$. Similar, market fragility happens only when the inter-dealer market friction is higher (i.e., $\lambda = 0$). As it becomes easier for customers to contact dealers, that is, as the

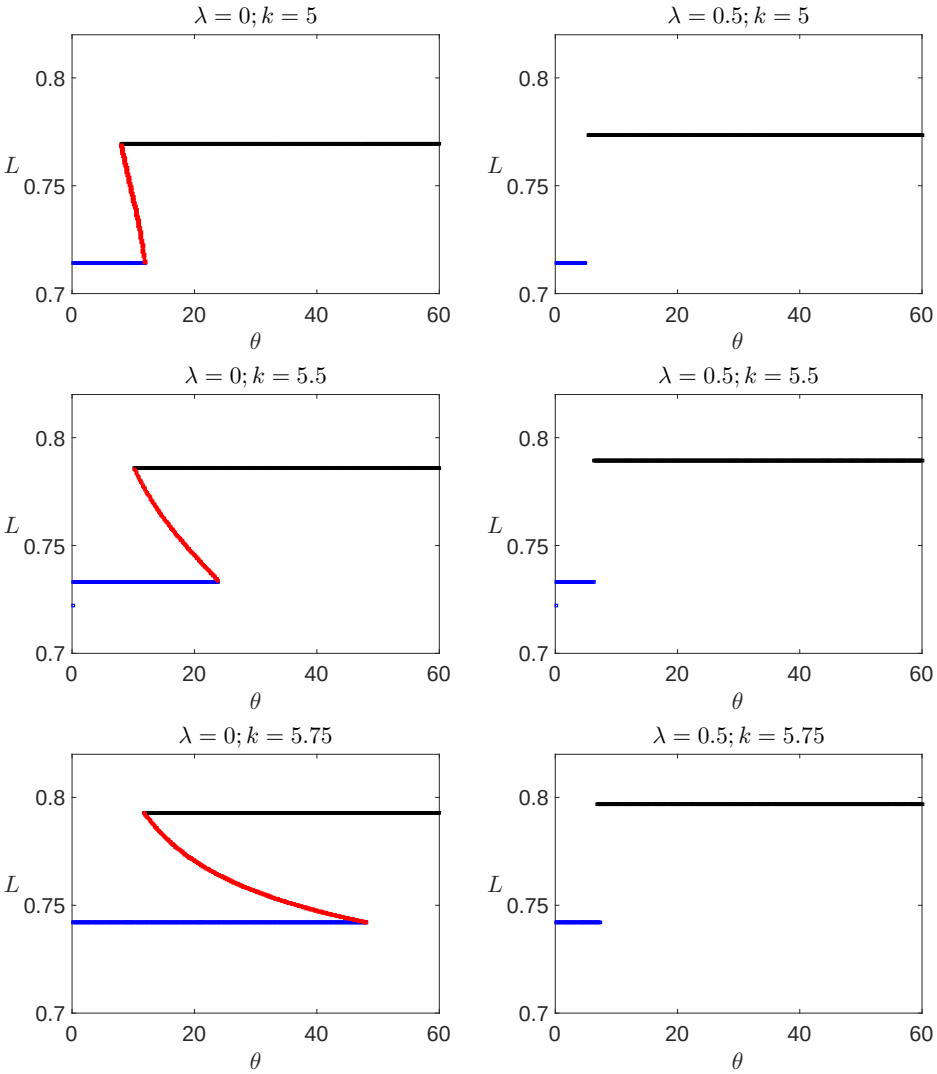

Figure 7: Equilibrium liquidity and fragility when λ and k vary

search friction becomes higher from 5 to 5.5 and then to 5.75, the equilibrium evolution has two additional interesting features.

First, equilibrium market liquidity becomes universally higher regardless the type of equilibrium. This is economically intuitive because a lower search friction makes it easier for customers to contact dealers, improving market liquidity.

However, perhaps surprisingly, market fragility in the sense of the possibility of multiplicity (happens when λ is close to 0) becomes higher as the search friction between customers and dealers becomes lower. However, this result is still economically intuitive due to the endogenous competition and coordination motives among dealers in our framework. When it becomes easier for customers to

contact the dealers, dealers complete more fiercely for order flows. Thus, the coordination motives among dealers to provide liquidity become higher, under the same level of inter-dealer market friction. This is broadly consistent with the pattern that a decline in trading frictions between customers and dealers in the recent decades may instead lead to a higher market fragility due to the escalated competition among financial intermediaries (e.g., Philippon, 2015).

6.4 Exit shocks, market fragility, and liquidity

We finally explore the impacts of search friction between customers and dealers on market fragility and liquidity across different parameter regions. The results are illustrated in Figure 8 below.

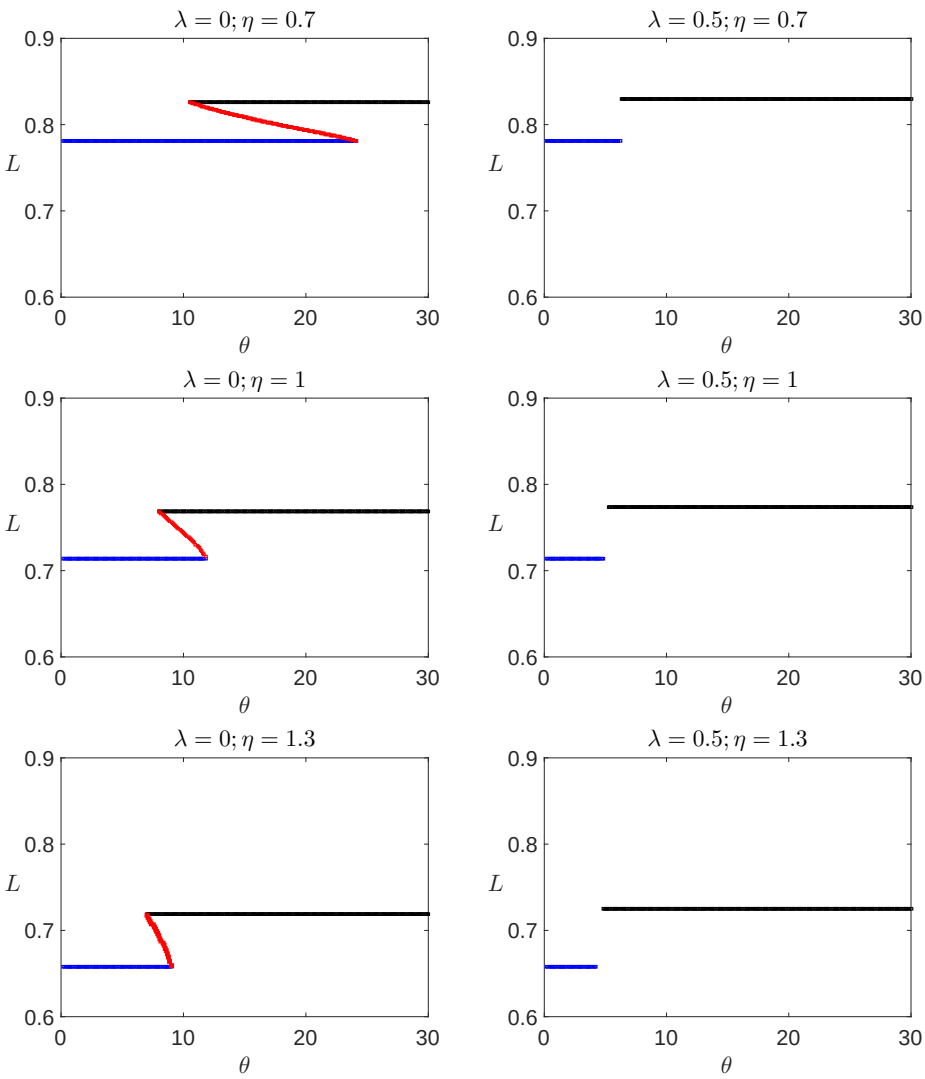

Figure 8: Equilibrium liquidity and fragility when λ and η vary

Here, we vary the inter-dealer market friction λ and the exit shock parameter η between customers and dealers, under the same inventory schedule $c = 0.1$ and $\rho = 2.5$, and search friction parameter $k = 5$. Similar, market fragility happens only when the inter-dealer market friction is higher (i.e., $\lambda = 0$).

As the exit shock parameter becomes higher from 0.7 to 1 and then to 1.3, equilibrium market liquidity becomes universally lower regardless the type of equilibrium due to the higher possibility of customer exiting without successfully finding a dealer. More interestingly, market fragility in the sense of the possibility of multiplicity (happens when λ is close to 0) becomes lower as the exit shock becomes larger. The intuition still comes from the endogenous competition and coordination motives among dealers in our framework. When the customers are able to attempt contacting the dealers for a longer time, dealers complete more fiercely for order flows. Thus, the coordination motives among dealers to provide liquidity become higher, under the same level of inter-dealer market friction.

7 Conclusion

In this paper, we study the coordination of intermediation in a dynamic intermediated asset market where dealers' participation and inventory holdings are endogenous. We show that an inter-dealer market may endogenously emerge, which further leads to endogenous coordination motives in dealers' inventory holding decisions. In an equilibrium where the inter-dealer market is active (inactive), the intermediation chains are longer (shorter), dealers hold a higher (lower) level of inventory on average, and they provide more (less) liquidity. Multiplicity may arise, suggesting the possibility of a shutdown of the inter-dealer market and liquidity drop even without a fundamental shock. The empirical implications are consistent with historical and recent developments in various over-the-counter markets.

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Appendix

A Derivation of the equilibrium with interdealer market

In any steady state, the amount of good inflow to the dealer sector equals to that of the good outflow, which implies

$$n - \eta y = n - \eta x \,, \tag{25-1}$$

and also

$$x \cdot \alpha \cdot \frac{\mu_1 + \mu_2}{\mu} = n - x\eta \,, \tag{25-2}$$

$$y \cdot \alpha \cdot \frac{\mu_1 + \mu_2}{\mu} = n - y\eta \,. \tag{25-3}$$

(25-1), (25-2), and (25-3) imply

$$\mu_0 = \mu_2 \,. \tag{25-4}$$

Outflow from μ_0 = inflow to μ_0, i.e.,

$$\mu_0(\beta \frac{y}{x+y} + \lambda \frac{\mu_2}{\mu}) = \mu_1 \beta \frac{x}{x+y} \,. \tag{25-5}$$

By (25-1), we have

$$\frac{\beta}{2}\mu_0 + \lambda \frac{\mu_0 \mu_2}{\mu} = \frac{\beta}{2}\mu_1 \,. \tag{25-6}$$

Note that (25-6) is also the "inflow - outflow" identity for μ_2, making the "inflow - outflow" identity of μ_1 redundant.

By (25-4), let

$$\mu_0 = \mu_2 = t \,, \cdot \mu \tag{26-1}$$

then by (25-6), we have

$$\frac{\beta}{2} t\mu + \lambda t^2 \mu = \frac{\beta}{2}(1 - 2t)\mu \,, \tag{26-2}$$

i.e.,

$$\lambda t^2 + \frac{3}{2}\beta t - \frac{\beta}{2} = 0 \,. \tag{26-3}$$

It has one negative root and a positive one, therefore we choose the positive one, i.e.,

$$t = \frac{-\frac{3}{2}\beta + \sqrt{\frac{9}{4}\beta^2 + 2\lambda\beta}}{2\lambda}\,. \tag{26-4}$$

Plug "$t = \frac{1}{3}$" into (26-3), then $LHS = \frac{\lambda}{9} \geqslant 0\,.$

Hence,

$$t \in [0, \frac{1}{3}]\,. \tag{26-5}$$

In particular,

$$\begin{cases} t = 0 & \text{if } \lambda \to \infty\,, \\ t = \frac{1}{3} & \text{if } \lambda = 0\,. \end{cases} \tag{26-6}$$

Differentiating (26-3), we have

$$2\lambda t dt + \frac{3}{2} t d\beta + \frac{3}{2}\beta dt - \frac{d\beta}{2} = 0,$$

i.e.,

$$(\frac{3\beta}{2} + 2\lambda t)\frac{dt}{d\beta} = \frac{1}{2}(1 - 3t)\,. \tag{26-7}$$

Therefore, if $\lambda > 0$, $t < \frac{1}{3}$ and thus

$$\frac{dt}{d\beta} > 0\,. \tag{26-8}$$

Below we uniquely pin down all the measures for a given μ (thus $\lambda = \lambda(\mu)$ is also given).

Note that $\alpha \cdot \underbrace{(x + y)}_{=2x} = \beta \cdot \mu$ is also an accounting identity. Hence,

$$x = \frac{\beta\mu}{2\alpha}\,. \tag{26-9}$$

(26-9), (26-1), and (25-2) imply

$$\frac{\mu}{2}\beta(1 - t) = n - \eta \cdot x\,. \tag{26-10}$$

By (**??**),

$$\beta(1 - t) = 2(\lambda t^2 + \beta t)\,.$$

Because $\frac{dt}{d\beta} > 0$, $\lambda t^2 + \beta t$ is increasing in β, and $\beta(1-t)$ is increasing in β.

Let $m(z,\mu)$ be the primitive matching function, where $m_z \geqslant 0$, $m_\mu \geqslant 0$, $\alpha = \frac{m(z,\mu)}{z}$ is decreasing in z, and $\beta = \frac{m(z,\mu)}{\mu}$ is decreasing in μ. Hence,

$$\beta(2x,\mu) = \frac{m(2x,\mu)}{\mu}\,, \tag{27-1}$$

and (27-1) is increasing in x, LHS of (26-10) is increasing in x, and RHS of (26-10) is decreasing in x. Hence $x = x(\mu)$ is uniquely determined by μ, then β is determined by (27-1), α is determined by (26-9), t is determined by (26-4), and μ_0 and μ_2 are determined by (26-1), thus $\mu_1 = (1-2t)\mu$.

Now consider the value functions in steady state for a given μ.

$$(\eta + r)V_b = \alpha[\underbrace{\frac{\mu_2}{\mu}}_{=t}\frac{1}{2}(\theta + V_1 - V_b - V_2) + \underbrace{\frac{\mu_1}{\mu}}_{=1-2t}\frac{1}{2}(\theta + V_0 - V_b - V_1)]\,, \tag{27-2}$$

$$(\eta + r)V_s = \alpha[\underbrace{\frac{\mu_0}{\mu}}_{=t}\frac{1}{2}(V_1 - V_0 - V_s) + \underbrace{\frac{\mu_1}{\mu}}_{=1-2t}\frac{1}{2}(V_2 - V_1 - V_s)]\,, \tag{27-3}$$

$$rV_1 = \beta[\underbrace{\frac{x}{x+y}}_{=\frac{1}{2}}\frac{1}{2}(\theta + V_0 - V_b - V_1) + \underbrace{\frac{y}{x+y}}_{=\frac{1}{2}}\frac{1}{2}(V_2 - V_1 - V_s)] - c_1\,, \tag{27-4}$$

$$rV_0 = \beta\underbrace{\frac{y}{x+y}}_{=\frac{1}{2}}\frac{1}{2}(V_1 - V_0 - V_s) + \lambda\underbrace{\frac{\mu_2}{\mu}}_{=t}\frac{1}{2}(2V_1 - V_0 - V_2) - c_0\,, \tag{27-5}$$

$$rV_2 = \beta\underbrace{\frac{x}{x+y}}_{=\frac{1}{2}}\frac{1}{2}(\theta + V_1 - V_b - V_2) + \lambda\underbrace{\frac{\mu_0}{\mu}}_{=t}\frac{1}{2}(2V_1 - V_0 - V_2) - c_2\,. \tag{27-6}$$

(27-2) holds if and only if

$$(\eta + r)V_b = \alpha[\frac{1}{2}(\theta - V_b)(1-t) - t\frac{1}{2}(V_2 - V_1) - (1-2t)\frac{1}{2}(V_1 - V_0)]\,. \tag{28-1}$$

(27-3) holds if and only if

$$(\eta + r)V_s = \alpha[-\frac{1}{2}V_s(1-t) + t\frac{1}{2}(V_1 - V_0) + (1-2t)\frac{1}{2}(V_2 - V_1)]\,. \tag{28-2}$$

(27-4) holds if and only if

$$rV_1 = \beta[\frac{1}{2}(\theta - V_b)\frac{1}{2} - \frac{1}{2}V_s\frac{1}{2} - \frac{1}{2}(V_1 - V_0)\frac{1}{2} + \frac{1}{2}(V_2 - V_1)\frac{1}{2}] - c_1 \,. \tag{28-3}$$

(27-5) holds if and only if

$$rV_0 = -\frac{\beta}{2}\frac{1}{2}V_s + \frac{1}{2}(V_1 - V_0)(\frac{\beta}{2} + \lambda t) - \frac{1}{2}(V_2 - V_1)\lambda t - c_0 \,. \tag{28-4}$$

(27-6) holds if and only if

$$rV_2 = \frac{1}{2}(\theta - V_b)\frac{\beta}{2} - \frac{1}{2}(V_2 - V_1)(\frac{\beta}{2} + \lambda t) + \frac{1}{2}(V_1 - V_0)\lambda t - c_2 \,. \tag{28-5}$$

(28-3) − (28-4) implies

$$r(V_1 - V_0) = \frac{\beta}{2}\frac{1}{2}(\theta - V_b) - \frac{1}{2}(V_1 - V_0)(\beta + \lambda t) + \frac{1}{2}(V_2 - V_1)(\frac{\beta}{2} + \lambda t) - c_1 + c_0 \,,$$

i.e.,

$$\frac{1}{2}(V_1 - V_0)[2r + \beta + \lambda t] = \frac{\beta}{2}\frac{1}{2}(\theta - V_b) + \frac{1}{2}(V_2 - V_1)(\frac{\beta}{2} + \lambda t) - c_1 + c_0 \,. \tag{28-6}$$

(28-5) − (28-3) implies

$$r(V_2 - V_1) = \frac{\beta}{2}\frac{1}{2}V_s - \frac{1}{2}(V_2 - V_1)(\beta + \lambda t) + \frac{1}{2}(V_1 - V_0)(\frac{\beta}{2} + \lambda t) - c_2 + c_1$$

i.e.,

$$\frac{1}{2}(V_2 - V_1)[2r + \beta + \lambda t] = \frac{\beta}{2}\frac{1}{2}V_s + \frac{1}{2}(V_1 - V_0)(\frac{\beta}{2} + \lambda t) - c_2 + c_1 \,. \tag{28-7}$$

(28-1) implies

$$V_b = (\eta + r + \frac{\alpha}{2}(1 - t))^{-1}\alpha[\frac{1}{2}\theta(1 - t) - \frac{1}{2}(V_2 - V_1)t - \frac{1}{2}(V_1 - V_0)(1 - 2t)] \,. \tag{29-1}$$

(28-2) implies

$$V_s = (\eta + r + \frac{\alpha}{2}(1 - t))^{-1}\alpha[\frac{1}{2}(V_1 - V_0) + \frac{1}{2}(V_2 - V_1)(1 - 2t)] \,. \tag{29-2}$$

(28-6) and (29-1) imply

$$\begin{aligned}\frac{1}{2}(V_1-V_0)[2r+\beta+\lambda t] =& \frac{\beta}{2}\frac{1}{2}[\theta-\frac{\frac{\alpha}{2}\theta(1-t)-\frac{\alpha}{2}t(V_2-V_1)-\frac{\alpha}{2}(1-2t)(V_1-V_0)}{\eta+r+\frac{\alpha}{2}(1-t)}]\\ &+\frac{1}{2}(V_2-V_1)(\frac{\beta}{2}+\lambda t)-c_1+c_0\\ =&\frac{\beta}{2}\frac{1}{2}\frac{(\eta+r)\theta+\alpha t\frac{1}{2}(V_2-V_1)+\alpha(1-2t)\frac{1}{2}(V_1-V_0)}{\eta+r+\frac{\alpha}{2}(1-t)}\\ &+\frac{1}{2}(V_2-V_1)(\frac{\beta}{2}+\lambda t)-c_1+c_0\,,\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{2}(V_1-V_0)[2r+\underbrace{\beta+\lambda t-\frac{\beta}{2}\frac{\alpha(1-2t)}{2(\eta+r)+\alpha(1-t)}}_{=B+\frac{\beta(\eta+r)}{2(\eta+r)+\alpha(1-t)}=B+Q}]\\ =&\frac{1}{2}(V_2-V_1)[\underbrace{\beta+\lambda t+\frac{\beta}{2}\frac{\alpha t}{2(\eta+r)+\alpha(1-t)}}_{=B}]\\ &+\frac{\beta(\eta+r)\frac{\theta}{2}}{2(\eta+r)+\alpha(1-t)}-c_1+c_0\,,\end{aligned} \tag{29-3}$$

i.e.,

$$\frac{1}{2}(V_1-V_0)[2r+B+Q]=\frac{1}{2}(V_2-V_1)B+\frac{\theta}{2}Q-c_1+c_0\,. \tag{29-4}$$

(28-7) and (29-2) imply

$$\begin{aligned}&\frac{1}{2}(V_2-V_1)[2r+\beta+\lambda t]\\ =&\frac{\beta}{2}\frac{1}{2}\frac{\alpha t\frac{1}{2}(V_1-V_0)+\alpha(1-2t)\frac{1}{2}(V_2-V_1)}{\eta+r+\frac{\alpha}{2}(1-t)}\\ &+\frac{1}{2}(V_1-V_0)(\frac{\beta}{2}+\lambda t)-c_2+c_1\,,\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{2}(V_2-V_1)[2r+\beta+\lambda t-\frac{\beta}{2}\frac{\alpha(1-2t)}{2(\eta+r)+\alpha(1-t)}]\\ =&\frac{1}{2}(V_1-V_0)(\frac{\beta}{2}+\lambda t+\frac{\beta}{2}\frac{\alpha t}{2(\eta+r)+\alpha(1-t)})-c_2+c_1\,,\end{aligned} \tag{30-1}$$

i.e.,

$$\frac{1}{2}(V_2 - V_1)[2r + B + Q] = \frac{1}{2}(V_1 - V_0)B - c_2 + c_1 \,. \tag{30-2}$$

(29-4) − (30-2) implies

$$\frac{1}{2}(2V_1 - V_2 - V_0)[2r + B + Q] = -\frac{1}{2}(2V_1 - V_2 - V_0)B + \frac{\theta}{2}Q + \underbrace{c_2 + c + 0 - 2c_1}_{\geqslant 0} \,,$$

i.e.,

$$\frac{1}{2}(2V_1 - V_2 - V_0)[2r + 2B + Q] = \frac{\theta}{2}Q + c_2 + c_0 - 2c_1 \,. \tag{30-3}$$

Because $B > 0$ and $Q > 0$, $2V_1 - V_2 - V_0 > 0$. Therefore, once $\mu_2 > 0$, inter-dealer trading happens.

(30-2) and (30-3) imply

$$\frac{1}{2}(V_2 - V_1)[2r + B + Q] = B\big[\frac{1}{2}(V_2 - V_1) + \frac{\frac{\theta}{2}Q + c_2 + c_0 - 2c_1}{2r + 2B + Q}\big] - c_2 + c_1 \,.$$

Therefore,

$$\frac{1}{2}(V_2 - V_1) = (2r + Q)^{-1}\big[\frac{B}{2r + 2B + Q}(\frac{Q}{2} + c_2 + c_0 - 2c_1) - c_2 + c_1\big] \,. \tag{31-1}$$

(31-1) and (29-4) imply

$$\frac{1}{2}(V_1 - V_0) = \frac{B}{2r + B + Q}\frac{1}{2}(V_2 - V_1) + \frac{\frac{\theta}{2}Q - c_1 + c_0}{2r + B + Q} \,, \tag{31-2}$$

i.e.,

$$\frac{1}{2}(V_1 - V_0)[2r + B + Q] = B\big[\frac{1}{2}(V_1 - V_0) - \frac{\frac{\theta}{2}Q + c_2 + c_0 - 2c_1}{2r + 2B + Q}\big] + \frac{\theta}{2}Q - c_1 + c_0 \,.$$

Therefore,

$$\frac{1}{2}(V_1 - V_0) = (2r + Q)^{-1}\big[\frac{-B}{2r + 2B + Q}(\frac{\theta}{2}Q + c_2 + c_0 - 2c_1) + \frac{\theta}{2}Q - c_1 + c_0\big] \,, \tag{31-3}$$

i.e.,

$$\frac{1}{2}(V_1 - V_0) = (2r + Q)^{-1}\frac{(2r + B + Q)(\frac{\theta}{2}Q - c_1 + c_0) - B(c_2 - c_1)}{2r + 2B + Q} \,. \tag{31-4}$$

Similarly, (31-1) can be rewritten as

$$\frac{1}{2}(V_2 - V_1) = (2r + Q)^{-1} \frac{B(\frac{\theta}{2}Q - c_1 + c_0) - (2r + B + Q)(c_2 - c_1)}{2r + 2B + Q}. \tag{31-5}$$

Note the symmetry between (31-4) and (31-5).

(27-3) implies

$$\begin{aligned} &\alpha(1-2t)\frac{1}{2}(V_2 - V_1 - V_s) = (\eta + r + \frac{\alpha t}{2})V_s - \alpha t \frac{1}{2}(V_1 - V_0) \\ =&\frac{2(\eta + r) + \alpha t}{2(\eta + r) + \alpha(1-t)}\left[\alpha t \frac{1}{2}(V_1 - V_0) + \alpha(1-2t)\frac{1}{2}(V_2 - V_1)\right] - \alpha t \frac{1}{2}(V_1 - V_0) \\ =&\frac{-\alpha(1-2t)\alpha t \frac{1}{2}(V_1 - V_0) + [2(\eta + r) + \alpha t]\alpha(1-2t)\frac{1}{2}(V_2 - V_1)}{2(\eta + r) + \alpha(1-t)}. \end{aligned} \tag{32-1}$$

To support the equilibrium with $\mu_2 > 0$, we need type "1" dealer to be willing to trader with the seller, which is equivalent to

$$V_2 - V_1 - V_s \geqslant 0. \tag{32-2}$$

Note that $t \in [0, \frac{1}{3}]$. Therefore, $\alpha(1-2t) > 0$ and $2(\eta + r) + \alpha(1-t) > 0$.

Hence, (32-2) if and only if $[2(\eta + r) + \alpha t]\frac{1}{2}(V_2 - V_1) - \alpha t \frac{1}{2}(V_1 - V_0) \geqslant 0$, which is equivalent to

$$2(\eta + r)\frac{1}{2}(V_2 - V_1) \geqslant \alpha t \frac{1}{2}(2V_1 - V_2 - V_0), \tag{32-3}$$

and by (31-5) and (30-3), we have

$$\frac{2(\eta + r)}{2r + Q} \frac{B(\frac{\theta}{2}Q - c_1 + c_0) - (2r + B + Q)(c_2 - c_1)}{2r + 2B + Q} \geqslant \frac{\alpha t[\frac{\theta}{2}Q + c_2 + c_0 - 2c_1]}{2r + 2B + Q}. \tag{32-4}$$

Since $B > 0$ and $Q > 0$, we also have

$$2(\eta + r)[B(\frac{\theta}{2}Q - c_1 + c_0) - (2r + B + Q)(c_2 - c_1)] \geqslant (2r + Q)\alpha t[\frac{\theta}{2}Q + c_2 + c_0 - 2c_1]. \tag{32-5}$$

(32-5) holds if and only if

$$2(\eta + r)[B(\frac{\theta}{2}Q + c_2 + c_0 - 2c_1) - (2r + 2B + Q)(c_2 - c_1)] \geqslant (2r + Q)\alpha t[\frac{\theta}{2}Q + c_2 + c_0 - 2c_1], \tag{33-1}$$

which is equivalent to

$$[\frac{\theta}{2}Q + c_2 + c_0 - 2c_1][2(\eta + r)B - (2r + Q)\alpha t] \geqslant 2(\eta + r)(c_2 - c_1)(2r + 2B + Q)\,. \tag{33-2}$$

Now we need to know if

$$2(\eta + r)B - (2r + Q)\alpha t > 0\,. \tag{33-3}$$

Intuitively, this should be true, since the higher θ is, the easier to support an equilibrium with $\mu_2 > 0$.

Plugging the expression of B and Q into (33-3), we have

$$2(\eta + r)\big[\frac{\beta}{2} + \lambda t + \frac{\beta}{2}\frac{\alpha t}{2(\eta + r) + \alpha(1 - t)}\big] > \alpha t\big(2r + \frac{\beta(\eta + r)}{2(\eta + r) + \alpha(1 - t)}\big)$$

, i.e.,

$$\beta + 2\lambda t + \frac{\beta\alpha t}{2(\eta + r) + \alpha(1 - t)} > \frac{2r}{\eta + r}\alpha t + \frac{\beta\alpha t}{2(\eta + r) + \alpha(1 - t)}$$

i.e.,

$$(\frac{2r}{\eta + r}\alpha - 2\lambda)t < \beta\,. \tag{33-4}$$

Note that (33-4) is a condition of steady state dynamics, in which α, λ, t, and β are uniquely pinned down by μ. Since $\theta > 0$, $Q > 0$, $c_2 + c_0 - 2c_1 \geqslant 0$, and RHS of (33-2) is strictly positive, (33-2) fails if (33-4) fails.

Below we derive rV_0.

(27-5) implies

$$
\begin{aligned}
rV_0 =& \frac{\beta}{2}[\frac{1}{2}(V_1 - V_0) - \frac{1}{2}V_s] + \lambda t\frac{1}{2}(2V_1 - V_0 - V_2) - c_0 \\
=& \frac{\beta}{2}\frac{[2(\eta + r) + \alpha(1 - 2t)]\frac{1}{2}(V_1 - V_0) - \alpha(1 - 2t)\frac{1}{2}(V_2 - V_1)}{2(\eta + r) + \alpha(1 - t)} \\
& + \lambda t\frac{1}{2}(2V_1 - V_0 - V_2) - c_0 \\
=& \frac{\beta}{2}\frac{2(\eta + r)\frac{1}{2}(V_1 - V_0) + \alpha(1 - 2t)\frac{1}{2}(2V_1 - V_0 - V_2)}{2(\eta + r) + \alpha(1 - t)} \\
& + \lambda t\frac{1}{2}(2V_1 - V_0 - V_2) - c_0 \\
=& \underbrace{\frac{\beta(\eta + r)}{2(\eta + r)} + \alpha(1 - t)}_{=Q}\frac{1}{2}(V_1 - V_0) \\
& + \frac{1}{2}(2V_1 - V_0 - V_2)\underbrace{\frac{\frac{\beta}{2}\alpha(1 - 2t) + \lambda t[2(\eta + t) + \alpha(1 - t)]}{2(\eta + r) + \alpha(1 - t)}}_{=\lambda t + \frac{\beta}{2}\frac{\alpha(1-2t)}{2(\eta+r)+\alpha(1-t)}} - c_0 \\
=& \frac{1}{2}(V_1 - V_0)Q + \frac{1}{2}(2V_1 - V_0 - V_2)[\beta + 2\lambda t - B - Q] - c_0 \,.
\end{aligned}
\tag{34-1}
$$

(34-1), (31-4), and (30-3) imply

$$
\begin{aligned}
rV_0 =& \frac{Q}{2r + Q}\frac{(2r + B + Q)(\frac{\theta}{2}Q - c_1 + c_0) - B(c_2 - c_1)}{2r + 2B + Q} \\
& + \frac{[\frac{\theta}{2}Q + c_2 + c_0 - 2c_1][\beta + 2\lambda t - B - Q]}{2r + 2B + Q} - c_0 \\
=& \frac{Q}{2r + Q}\Big[\frac{(2r + B + Q)(\frac{\theta}{2}Q + c_2 + c_0 - 2c_1)}{2r + 2B + Q} - (c_2 - c_1)\Big] \\
& + \frac{\beta + 2\lambda t - B - Q}{2r + 2B + Q}(\frac{\theta}{2}Q + c_2 + c_0 - 2c_1) - c_0 \\
=& (\frac{\theta}{2}Q + c_2 + c_0 - 2c_1)\frac{Q(2r + B + Q) + (2r + Q)(\beta + 2\lambda t - B - Q)}{(2r + Q)(2r + 2B + Q)} \\
& - \frac{Q}{2r + Q}(c_2 - c_1) - c_0 \\
=& (\frac{\theta}{2}Q + c_2 + c_0 - 2c_1)\frac{(\beta + 2\lambda t)(2r + Q) - 2rB}{(2r + Q)(2r + 2B + Q)} - \frac{Q}{2r + Q}(c_2 - c_1) - c_0 \,.
\end{aligned}
\tag{35-1}
$$

Note that

$$
(\beta + 2\lambda t)(2r + Q) - 2rB = \underbrace{(\beta + 2\lambda t - \beta)}_{=\lambda t + \frac{\beta}{2}\frac{2(\eta+r)+\alpha(1-2t)}{2(\eta+r)+\alpha(1-t)} > 0} 2r + (\beta + 2\lambda t)Q \,. \tag{35-2}
$$

Hence $(\beta + 2\lambda t)(2r + Q) - 2rB > 0$. Note that once μ is given, θ, β, λ, t, and B are all determined.

Therefore, there exists $\hat{\theta}_0(\mu)$ such that $V_0 \geqslant 0$ for all $\theta \geqslant \hat{\theta}_0(\mu)$, where $\hat{\theta}_0(\mu)$ is defined such that as $rV_0 = 0$.

Now we compare $\hat{\theta}_0(\mu)$ to $\hat{\theta}_2(\mu)$ when (33-4) holds. Note that by definition

$$\frac{\hat{\theta}_2}{2}Q + c_2 + c_0 - 2c_1 = \frac{2(\eta + r)(2r + 2B + Q)(c_2 - c_1)}{2(\eta + r)B - (2r + Q)\alpha t}\,. \tag{36-1}$$

Plugging it into (35-1), we have

$$\begin{aligned} rV_0(\hat{\theta}_2) &= \frac{2(\eta + r)(2r + 2B + Q)(c_2 - c_1)}{2(\eta + r)B - (2r + Q)\alpha t}\,\frac{(\beta + 2\lambda t)(2r + Q) - 2rB}{(2r + Q)(2r + 2B + Q)} \\ &= (c_2 - c_1)\underbrace{\Big[-1 + \frac{2(\eta + r)(\beta + 2\lambda t) - 2r\alpha t}{2(\eta + r)B - (2r + Q)\alpha t}\Big]}_{>0} - c_0\,. \end{aligned} \tag{36-2}$$

Note that $-1 + \frac{2(\eta+r)(\beta+2\lambda t)-2r\alpha t}{2(\eta+r)B-(2r+Q)\alpha t}$ is determined by μ, and not dependent on c_0, c_1, or c_2. Hence, for some values of μ, it could be $rV_0(\hat{\theta}_2) > 0$, and for some values of μ, $rV_0(\hat{\theta}_2) < 0$.

B Derivation of the equilibrium without inter-dealer market

Now consider the equilibrium in which $\mu_2 = 0$, $\eta > 0$, $\delta_s = \delta_D = 0$, $\delta_b = 1$, and $\theta > 0$.

Again, $n - \eta x = n - \eta y$ implies

$$x = y\,, \tag{38-1}$$

$$x\alpha\frac{\mu_1}{\mu} = n - \eta x\,, \tag{38-2}$$

$$y\alpha\frac{\mu_0}{\mu} = n - \eta x\,. \tag{38-3}$$

By (38-1), (38-2), and (38-3), we have

$$\mu_0 = \mu_1 = \frac{\mu}{2}\,. \tag{38-4}$$

Note that (38-1) and (38-4) make the steady state conditions for μ_0 and μ_1 automatically hold.

Note that

$$\beta = \frac{m(2x, \mu)}{\mu}\,, \tag{38-5}$$

$$\alpha = \frac{m(2x,\mu)}{2x}\,. \tag{38-6}$$

(38-1) and (38-2) imply

$$\alpha = \frac{2(n-\eta x)}{x} = 2(\frac{n}{x} - \eta)\,. \tag{38-7}$$

(38-5), (38-6), and (38-7) imply

$$\frac{m(2x,\mu)}{2x} = 2(\frac{n}{x} - \eta), \tag{38-8}$$

which is equivalent to

$$m(2x,\mu) = 4(n-\eta x)\,. \tag{38-9}$$

Note that LHS is increasing in x, and RHS decreasing in x. Hence, for any $\mu > 0$, there exists a unique $x = x(\mu)$ to solve (38-9), then α and β are all determined by μ. Note that $\lambda(\mu)$ is by definition determined by μ. "λ" will be used at the end to verify the equilibrium, although not in the characterization.

Value functions under a given $\mu > 0$.

$$(\eta + r)V_b = \alpha\frac{\mu_1}{\mu}\frac{1}{2}(\theta + V_0 - V_b - V_1) = \frac{\alpha}{2}\big[\frac{1}{2}(\theta - V_b) - \frac{1}{2}(V_1 - V_0)\big]\,, \tag{39-1}$$

$$(\eta + r)V_s = \alpha\frac{\mu_0}{\mu}\frac{1}{2}(V_1 - V_0 - V_s) = \frac{\alpha}{2}\big[\frac{1}{2}(V_1 - V_0) - \frac{1}{2}V_s\big]\,, \tag{39-2}$$

$$rV_0 = \beta\frac{y}{x+y}\frac{1}{2}(V_1 - V_0 - V_s) - c_0 = \frac{\beta}{2}\big[\frac{1}{2}(V_1 - V_0) - \frac{1}{2}V_s\big] - c_0\,, \tag{39-3}$$

$$rV_1 = \beta\frac{x}{x+y}\frac{1}{2}(\theta + V_0 - V_b - V_1) - c_1 = \frac{\beta}{2}\big[\frac{1}{2}(\theta - V_b) - \frac{1}{2}(V_1 - V_0)\big] - c_1\,. \tag{39-4}$$

(39-1) and (39-4) imply

$$rV_1 = \frac{\beta(\eta + r)V_b}{\alpha} - c_1\,. \tag{39-5}$$

(39-2) and (39-3) imply

$$rV_0 = \frac{\beta(\eta + r)V_s}{\alpha} - c_0\,. \tag{39-6}$$

(39-4) − (39-3) imply

$$\begin{aligned} r(V_1 - V_0) &= -\beta\frac{1}{2}(V_1 - V_0) + \frac{\beta}{2}\big(\frac{1}{2}(\theta - V_b) + \frac{1}{2}V_s\big) - c_1 + c_0 \\ &= -\beta\frac{1}{2}(V_1 - V_0) + \frac{\beta}{4}\big[\theta + (V_2 - V_b)\big] - c_1 + c_0\,. \end{aligned} \tag{39-7}$$

(39-5) and (39-4) imply

$$r(V_1 - V_0) = \frac{\beta}{\alpha}(\eta + r)(V_b - V_s) - c_1 + c_0\,.$$

Therefore,

$$V_s - V_b = \frac{\alpha}{\beta(\eta + r)}\big[-r(V_1 - V_0) - c_1 + c_0\big]\,. \tag{40-1}$$

(39-7) and (40-1) imply

$$r(V_1 - V_0) = -\frac{\beta}{2}(V_1 - V_0) + \frac{\beta}{4}\theta + \frac{\alpha}{4(\eta + r)}\big[-r(V_1 - V_0) - c_1 + c_0\big] - c_1 + c_0\,.$$

Therefore,

$$V_1 - V_0 = \big[r + \frac{\beta}{2} + \frac{r\alpha}{4(\eta + r)}\big]^{-1}\big[\frac{\beta}{4}\theta - (c_1 - c_0)(1 + \frac{\alpha}{4(\eta + r)})\big]\,. \tag{40-2}$$

(39-6) implies

$$V_s = \frac{\alpha}{\beta(\eta + r)}(rV_0 + c_0)\,. \tag{40-3}$$

(40-3) and (39-3) imply

$$rV_0 = \frac{\beta}{4}(V_1 - V_0) - \frac{\beta}{4}V_s - c_0 = \frac{\beta}{4}(V_1 - V_0) - \frac{\alpha}{4(\eta + r)}(rV_0 + c_0) - c_0\,.$$

Therefore,

$$r\big[1 + \frac{\alpha}{4(\eta + r)}\big]V_0 = \frac{\beta}{4}(V_1 - V_0) - \big[\underbrace{1 + \frac{\alpha}{4(\eta + r)}}_{=D}\big]c_0\,. \tag{40-4}$$

Let $D = 1 + \frac{\alpha}{4(\eta+r)}$. Then (40-2) becomes

$$V_1 - V_0 = \frac{\frac{\beta}{4}\theta - D(c_1 - c_0)}{\frac{\beta}{2} + rD}\,. \tag{40-5}$$

(40-5) and (40-4) imply

$$rDV_0 = \frac{\beta}{4}\frac{\frac{\beta}{4}\theta - D(c_1 - c_0)}{\frac{\beta}{2} + rD} - Dc_0\,. \tag{41-1}$$

As a requirement for the equilibrium, we need $V_0 \geqslant 0$. By (41-1), (since $D > 0$,)

$$\frac{\beta}{4}\theta - D(c_1 - c_0) \geqslant \frac{[2\beta + 4rD]Dc_0}{\beta}\,,$$

i.e.,

$$\theta \geqslant \underbrace{8\beta^{-2}(\beta + 2rD)Dc_0 + 4\beta^{-1}D(c_1 - c_0)}_{=\hat{\theta}_1(\mu)}\,. \tag{41-2}$$

Note that (41-2) is sufficient to make all transactions happen.

In this equilibrium with $\mu_0 = 0$. To see this, note that $V_0 \geqslant 0$ implies $V_1 - V_0 - V_s > 0$, which by (39-2), implies $V_s > 0$.

To support this equilibrium where $\mu_2 =$, we also need that "1" dealers don't want to buy from sellers. Suppose that a zero-measure of "1" dealer considers deviation. We examine V_2 in this virtual environment.

$$rV_2 = \beta\underbrace{\frac{x}{x+y}}_{=\frac{1}{2}}\frac{1}{2}(\theta + V_1 - V_b - V_2) + \lambda\underbrace{\frac{\mu_0}{\mu}}_{=\frac{1}{2}}\frac{1}{2}(2V_1 - V_0 - V_2) - c_2\,. \tag{42-1}$$

Therefore,

$$V_2 = (r + \frac{\lambda}{4} + \frac{\beta}{4})^{-1}\big[\frac{\beta}{4}(\theta + V_1 - V_b) + \frac{\lambda}{4}(2V_1 - V_0) - c_2\big]\,.$$

By (39-4),

$$\frac{\beta}{4}(\theta + V_1 - V_b) + \frac{\lambda}{4}(2V_1 - V_0) = rV_1 + c_1 + \frac{\beta + \lambda}{4}(2V_1 - V_0)\,. \tag{42-2}$$

Therefore,

$$\begin{aligned} V_2 &= \frac{rV_1 + \frac{\beta+\lambda}{4}(2V_1 - V_0) + c_1 - c_2}{r + \frac{\lambda+\beta}{4}} \\ &= V_1 + \frac{\frac{\beta+\lambda}{4}(V_1 - V_0) + c_1 - c_2}{r + \frac{\lambda+\beta}{4}}\,. \end{aligned} \tag{42-3}$$

Hence,

$$V_2 - V_1 - V_s = \frac{\frac{\beta+\lambda}{4}(V_1 - V_0) + c_1 - c_2}{r + \frac{\lambda+\beta}{4}} - V_s\,. \tag{42-4}$$

(40-3) and (42-4) imply

$$V_2 - V_1 - V_s = \frac{\frac{\beta+\lambda}{4}(V_1 - V_0) + c_1 - c_2}{r + \frac{\lambda+\beta}{4}} - \frac{\alpha}{\beta(\eta + r)}(rV_0 + c_0)\,. \tag{43-1}$$

To support the equilibrium with $\mu_2 = 0$, we need $V_2 - V_1 - V_s \leqslant 0$, i.e.,

$$\frac{\frac{\beta+\lambda}{4}(V_1 - V_0) + c_1 - c_2}{r + \frac{\lambda+\beta}{4}} \leqslant \frac{\alpha}{\beta(\eta + r)}(rV_0 + c_0)\,. \tag{43-2}$$

By (40-5) and (41-1), we have

$$\frac{\frac{\beta+\lambda}{4}\frac{\frac{\beta}{4}\theta - D(c_1 - c_0)}{\frac{\beta}{2} + rD} + c_1 - c_2}{r + \frac{\lambda+\beta}{4}} \leqslant \frac{\alpha}{\beta(\eta + r)}\Big[\frac{\beta}{4D}\frac{\frac{\beta}{4}\theta - D(c_1 - c_0)}{\frac{\beta}{2} + rD}\Big]\,,$$

i.e.,

$$\frac{(\beta + \lambda)(V_1 - V_0) + 4(c_1 - c_2)}{4r + \lambda + \beta} \leqslant \frac{\alpha}{4(\eta + r) + \alpha}(V_1 - V_0)\,, \tag{43-3}$$

i.e.,

$$\Big[\frac{\beta + \lambda}{4r + \beta + \lambda} - \frac{\alpha}{4(\eta + r) + \alpha}\Big](V_1 - V_0) \leqslant \frac{4(c_2 - c_1)}{4r + \lambda + \beta}\,, \tag{43-4}$$

i.e.,

$$\frac{\beta + \lambda}{4r + \beta + \lambda} - \frac{\alpha}{4(\eta + r) + \alpha} = \frac{4(\eta + r)(\beta + \lambda) - 4r\alpha}{(4r + \beta + \lambda)[4(\eta + r) + \alpha]}\,.$$

Therefore, if $4(\eta + r)(\beta + \lambda) - 4r\alpha \leqslant 0$, then (43-4) always hold. If $4(\eta + r)(\beta + \lambda) - 4r\alpha > 0$, then by (40-2), we need θ smaller than or equal to some $\hat{\theta}_c$.

Let

$$\bar{\beta} = \lim_{\mu \to 0} \frac{m(2x, \mu)}{\mu} \in (0, \infty)\,. \tag{44-1}$$

We consider $\underline{\theta}$ such that for all $\theta > \underline{\theta}$, there cannot be an equilibrium with $\mu = 0$.

If $\mu = 0$, consider an infinitesimal dealer enters the market, then buyers and sellers have zero reservation value in trading, i.e., $V_b = V_s = 0$ since $\mu = 0$,

$$rV_0 = \frac{\bar{\beta}}{2}\frac{1}{2}(V_1 - V_0 - V_s) - c_0 = \frac{\bar{\beta}}{4}(V_1 - V_0) - c_0\,, \tag{44-2}$$

$$rV_1 = \frac{\bar{\beta}}{4}(\theta + V_0 - V_1) + \frac{\bar{\beta}}{4}(V_2 - V_1) - c_1\,, \tag{44-3}$$

$$rV_2 = \frac{\bar{\beta}}{4}(\theta + V_1 - V_2) - c_2 \,. \tag{44-4}$$

(44-4) − (44-3) implies

$$r(V_2 - V_1) = \frac{\bar{\beta}}{4}\big[(V_1 - V_0) - 2(V_2 - V_1)\big] - c_2 + c_1 \,.$$

Therefore,

$$V_2 - V_1 = (r + \frac{\bar{\beta}}{2})^{-1}\big[\frac{\bar{\beta}}{4}(V_1 - V_0) - c_2 + c_1\big] \,. \tag{44-5}$$

(44-3) − (44-2) implies

$$\begin{aligned} r(v_1 - V_0) &= \frac{\bar{\beta}}{4}\big[\theta - 2(V_1 - V_0) + V_2 - V_1\big] - c_1 + c_0 \\ &= \frac{\bar{\beta}}{4}\big[\theta - 2(V_1 - V_0) + \frac{\frac{\bar{\beta}}{4}(V_1 - V_0) - c_2 + c_1}{r + \frac{\bar{\beta}}{2}}\big] - c_1 + c_0 \,. \end{aligned}$$

Therefore,

$$V_1 - V_0 = (r + \frac{\bar{\beta}}{2} - \frac{\bar{\beta}}{4}\frac{1}{r + \frac{\bar{\beta}}{2}})^{-1}\big[\frac{\bar{\beta}}{4}\theta - \frac{c_2 - c_1}{r + \frac{\bar{\beta}}{2}}\frac{\bar{\beta}}{4} - c_1 + c_0\big] \,. \tag{44-6}$$

If $\bar{\beta} \to \infty$, then $V_0 \to \infty$ for all $\theta > 0$. Therefore, in an equilibrium with $\mu > 0$, the only question is whether $\mu_2 > 0$.

C Derivation of the mixed-strategy equilibria

Consider the case where $V_2 - V_1 - V_s = 0$ and once a seller and a type "1" dealer meet, they trade with probability $\phi \in [0, 1]$. Then the previously analyzed equilibria correspond to $\phi = 1$ or $\phi = 0$.

Again we still have $n - \eta y = n - \eta x$, which implies

$$x = y \,, \tag{45-1}$$

$$x\alpha\frac{\mu_1 + \mu_2}{\mu} = n - \eta y \,, \tag{45-2}$$

$$y\alpha\frac{\phi\mu_1 + \mu_0}{\mu} = n - \eta y \,. \tag{45-3}$$

(45-1), (45-2), and (45-3) together imply

$$\mu_1 + \mu_2 = \phi\mu_1 + \mu_0 \,, \tag{45-4}$$

i.e., $\mu_2 = \mu_0 - (1-\phi)\mu_1$.

Outflow from μ_0 = inflow to μ_0

$$mu_0(\beta\frac{y}{x+y} + \lambda\frac{\mu_2}{\mu}) = \mu_1\beta\frac{x}{x+y} \,. \tag{45-5}$$

Outflow from μ_2 = inflow to μ_2

$$\mu_2(\beta\frac{x}{x+y} + \lambda\frac{\mu_0}{\mu}) = \mu_1\beta\frac{\phi y}{x+y} \,. \tag{45-6}$$

Note that (45-6) and (45-5) imply the "inflow - outflow" identity for μ_1. Further, (45-1), (45-4), and (45-5) imply (45-6). Therefore, we just need (45-1), (45-4), and (45-5).

Let

$$\mu_0 = t\mu \,. \tag{46-1}$$

Therefore, (45-4) implies $\phi\mu_1 + \mu_0 = (1-t)\mu$, i.e.,

$$\mu_1 = \phi^{-1}(1-2t)\mu \tag{46-2}$$

Hence,

$$\mu_2 = \mu - \mu_0 - \mu_1 = [1 - t - \phi^{-1}(1-2t)]\mu \,. \tag{46-3}$$

(45-1), (46-1), (46-2), (46-3), and (45-5) imply

$$t\big(\frac{\beta}{2} + \lambda[1 - t - \phi^{-1}(1-2t)]\big) = \phi^{-1}(1-2t)\frac{\beta}{2} \,,$$

i.e.,

$$\phi\frac{\beta}{2}t + \phi\lambda t - \phi\lambda t^2 - \lambda t(1-2t) = \frac{\beta}{2}(1-2t) \,,$$

i.e.,

$$(2-\phi)\lambda t^2 + (\phi\frac{\beta}{2} + \phi\lambda - \lambda + \beta)t - \frac{\beta}{2} = 0\,,$$

i.e.,

$$(2-\phi)\lambda t^2 + \big((1+\frac{\phi}{2})\beta - (1-\phi)\lambda\big)t - \frac{\beta}{2} = 0\,. \tag{46-4}$$

Define that

$$\begin{aligned}\Delta &= \big[(1+\frac{\phi}{2})\beta - (1-\phi)\lambda\big]^2 + 2\beta(2-\phi)\lambda \\ &= (1+\frac{\phi}{2})^2\beta^2 + (1-\phi)^2\lambda^2 - 2(1+\frac{\phi}{2})\beta(1-\phi)\lambda + 2\beta(2-\phi)\lambda,\end{aligned}$$

where $-2(1+\frac{\phi}{2})\beta(1-\phi)\lambda + 2\beta(2-\phi)\lambda = 2\beta\lambda[2-\phi-1+\frac{\phi}{2}+\frac{\phi^2}{2}] = 2\beta\lambda[(1-\frac{\phi}{4})^2 + \frac{7}{16}\phi^2 > 0$.

Therefore, $\Delta > 0$ and

$$t = \frac{(1-\phi)\lambda - (1+\frac{\phi}{2})\beta + \sqrt{(1+\frac{\phi}{2})^2\beta^2 + (1-\phi)^2\lambda^2 + 2\beta\lambda(\frac{\phi^2}{2} - \frac{\phi}{2} + 1)}}{2(2-\phi)\lambda}\,. \tag{46-5}$$

Note that the negative solution is dropped since $t \in [0,1]$.

(46-5) implies

$$\lim_{\lambda\to\infty} t = 0 \tag{47-1}$$

when $\lambda = 0$.

(46-4) implies

$$t = \frac{\beta}{2(1+\frac{\phi}{2})\beta} = \frac{1}{2+\phi}\,. \tag{47-2}$$

Plugging $t = \frac{1}{2+\phi}$ into (46-4), we see

$$\begin{aligned}&(2-\phi)\lambda\frac{1}{(2+\phi)^2} + \frac{1+\frac{\phi}{2}}{2+\phi}\beta - \frac{1-\phi}{2+\phi}\lambda - \frac{\beta}{2} \\ =&\frac{\lambda}{2+\phi}\big[\frac{2-\phi}{2+\phi} - 1 + \phi\big] = \frac{\lambda\phi}{2+\phi}(-\frac{2}{2+\phi} + 1) \\ =&\frac{\lambda\phi}{2+\phi}\frac{\phi}{2+\phi} > 0 \quad (= 0 \text{ if } \phi = 0)\,.\end{aligned}$$

Hence,

$$t \in [0, \frac{1}{2+\phi}]\,. \tag{47-3}$$

Differentiating (46-4), we have

$$2(2-\phi)\lambda t dt + (1+\frac{\phi}{2})t d\beta + [(1+\frac{\phi}{2}\beta - (1-\phi)\lambda]dt = \frac{1}{2}d\beta\,.$$

Hence,

$$\underbrace{[2(2-\phi)\lambda t + (1+\frac{\phi}{2})\beta - (1-\phi)\lambda]}_{=\frac{\frac{\beta}{2}+(2-\phi)\lambda t^2}{t}>0}\frac{dt}{d\beta} = \underbrace{\frac{1}{2}[1-(2+\phi)t]}_{>0}\,.$$

Therefore,

$$\frac{dt}{d\beta} > 0\,. \tag{47-4}$$

Also, $\alpha(x+y) = \beta\mu$ and $x = y$ imply

$$x = \frac{\beta\mu}{2\alpha}\,. \tag{48-1}$$

(45-2), (46-1), and (48-1) impy

$$\frac{\beta\mu}{2\alpha}\alpha(1-t) = n - xy\,. \tag{48-2}$$

Note that (46-4) implies

$$\frac{\beta(1-t)}{2} = (2-\phi)\lambda t^2 + \big[\frac{1+\phi}{2}\beta - \lambda(1-\phi)\big]t\,. \tag{48-3}$$

Therefore,

$$\begin{aligned}\frac{d}{d\beta}\big[\frac{\beta}{2}(1-t)\big] =& 2(2-\phi)\lambda t\frac{dt}{d\beta} + \frac{1+\phi}{2}t + \big[\frac{1+\phi}{2}\beta - \lambda(1-\phi)\big]\frac{dt}{d\beta}\\ =& \big[\underbrace{2(2-\phi)\lambda t + \frac{1+\phi}{2}\beta - \lambda(1-\phi)}_{=\frac{\frac{\beta}{2}(1-t)+(2-\phi)\lambda t^2}{t}>0}\big]\underbrace{\frac{dt}{d\beta}}_{>0} + \underbrace{\frac{1+\phi}{2}t}_{>0}\,.\end{aligned} \tag{48-4}$$

Hence, $\frac{\beta(1-t)}{2}$ is increasing in β, since β is increasing in x. Thus, $\frac{\beta(1-t)}{2}$ is increasing in x. Therefore, LHS of (48-2) is increasing in x, and RHS decreasing in x.

Also note that (48-2) can be rewritten as

$$\frac{1}{2}m(2x,\mu)(1-t) = n - xy\,. \tag{48-5}$$

Hence, $x = 0$ implies that LHS $= 0 < n =$ RHS, and $x = \frac{n}{\eta}$ implies LHS $> 0 =$ RHS.

Therefore, once x is pinned down by (48-5), β is pinned down by $\beta = \frac{m(2x,\mu)}{\mu}$, α by $\beta = \frac{m(2x,\mu)}{2x}$, t by (46-5), and μ_0, μ_1, and μ_2 by (46-1), (46-2), and (46-3).

Finally, we use (27-2) to (27-6) to figure out V_b, V_s, V_0, V_1, and V_2, then check if $V_2 - V_1 - V_s = 0$ and adjust $\phi \in [0,1]$ accordingly. Note that when using (27-2) to (27-6), $\frac{\mu_2}{\mu} = 1 - t - \phi^{-1}(1-2t)$, and $\frac{\mu_1}{\mu} = \phi^{-1}(1-2t)$.